SHORTEST PATHS

Shortest paths in an edge-weighted digraph

Given an edge-weighted digraph, find the shortest path from s to t.

edge-weighted digraph

```
4->5  0.35
5->4  0.35
4->7  0.37
5->7  0.28
7->5  0.28
5->1  0.32
0->4  0.38
0->2  0.26
7->3  0.39
1->3  0.29
2->7  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```

What is the shortest Path from 0 to 6

Shortest path applications

http://en.wikipedia.org/wiki/Seam_carving

- Map routing.
- Seam carving.
- Robot navigation.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

Shortest path variants

Which vertices?

- Single source: from one vertex s to every other vertex.
- Single sink: from every vertex to one vertex t.
- Source-sink: from one vertex s to another t.
- All pairs: between all pairs of vertices.

Restrictions on edge weights?

- Nonnegative weights.
- Euclidean weights.
- Arbitrary weights.

Cycles?

- No directed cycles.
- No "negative cycles."

which variant?

Simplifying assumption. Shortest paths from s to each vertex v exist.

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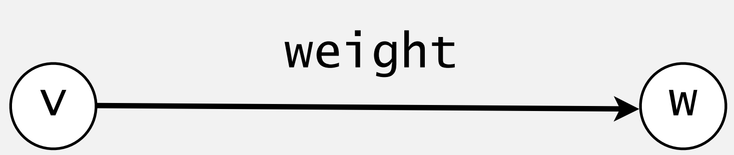
4.4 SHORTEST PATHS

- *APIs*
- *shortest-paths properties*
- *Dijkstra's algorithm*
- *edge-weighted DAGs*
- *negative weights*

Weighted directed edge API

public class DirectedEdge		
	DirectedEdge(int v, int w, double weight)	*weighted edge v→w*
int	from()	*vertex v*
int	to()	*vertex w*
double	weight()	*weight of this edge*
String	toString()	*string representation*



Idiom for processing an edge `e`: `int v = e.from(), w = e.to();`

Weighted directed edge: implementation in Java

Similar to `Edge` for undirected graphs, but a bit simpler.

```
public class DirectedEdge
{
   private final int v, w;
   private final double weight;

   public DirectedEdge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int from()
   {  return v;  }

   public int to()
   {  return w;  }

   public int weight()
   {  return weight; }
}
```

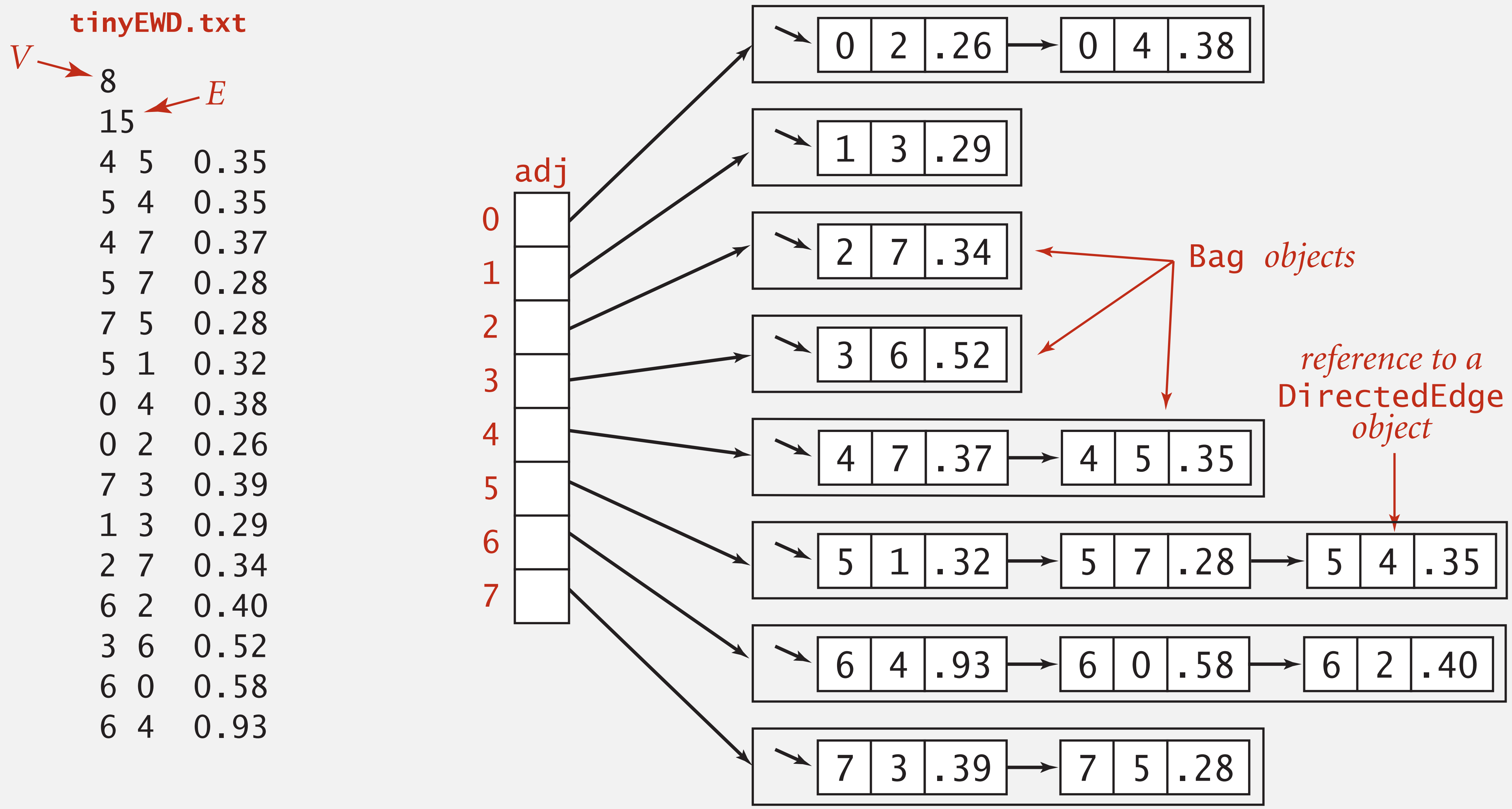
from() and to() replace either() and other()

Edge-weighted digraph API

public class	EdgeWeightedDigraph	
	EdgeWeightedDigraph(int V)	*edge-weighted digraph with V vertices*
	EdgeWeightedDigraph(In in)	*edge-weighted digraph from input stream*
void	addEdge(DirectedEdge e)	*add weighted directed edge e*
Iterable<DirectedEdge>	adj(int v)	*edges pointing from v*
int	V()	*number of vertices*
int	E()	*number of edges*
Iterable<DirectedEdge>	edges()	*all edges*
String	toString()	*string representation*

Conventions. Allow self-loops and parallel edges.

Edge-weighted digraph: adjacency-lists representation



Edge-weighted digraph: adjacency-lists implementation in Java

Same as `EdgeWeightedGraph` except replace `Graph` with `Digraph`.

```
public class EdgeWeightedDigraph
{
   private final int V;
   private final Bag<DirectedEdge>[] adj;

   public EdgeWeightedDigraph(int V)
   {
      this.V = V;
      adj = (Bag<DirectedEdge>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<DirectedEdge>();
   }

   public void addEdge(DirectedEdge e)
   {
      int v = e.from();
      adj[v].add(e);
   }

   public Iterable<DirectedEdge> adj(int v)
   {  return adj[v];  }
}
```

add edge e = v→w to only v's adjacency list

Single-source shortest paths API

Goal. Find the shortest path from s to every other vertex.

public class SP

	SP(EdgeWeightedDigraph G, int s)	*shortest paths from s in graph G*
double	distTo(int v)	*length of shortest path from s to v*
Iterable <DirectedEdge>	pathTo(int v)	*shortest path from s to v*
boolean	hasPathTo(int v)	*is there a path from s to v?*

```
SP sp = new SP(G, s);
for (int v = 0; v < G.V(); v++)
{
   StdOut.printf("%d to %d (%.2f): ", s, v, sp.distTo(v));
   for (DirectedEdge e : sp.pathTo(v))
      StdOut.print(e + "  ");
   StdOut.println();
}
```

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4.4 Shortest Paths

- *APIs*
- *shortest-paths properties*
- *Dijkstra's algorithm*
- *edge-weighted DAGs*
- *negative weights*

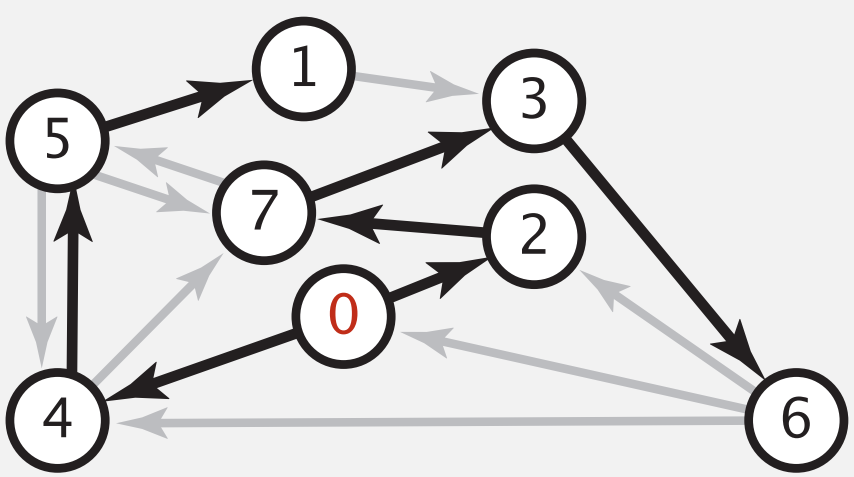
Data structures for single-source shortest paths

Goal. Find the shortest path from s to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- `distTo[v]` is length of shortest path from s to v.
- `edgeTo[v]` is last edge on shortest path from s to v.



shortest-paths tree from 0

	edgeTo[]		distTo[]
0	null		0
1	5->1	0.32	1.05
2	0->2	0.26	0.26
3	7->3	0.37	0.97
4	0->4	0.38	0.38
5	4->5	0.35	0.73
6	3->6	0.52	1.49
7	2->7	0.34	0.60

parent-link representation

Data structures for single-source shortest paths

Goal. Find the shortest path from s to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- `distTo[v]` is length of shortest path from s to v.
- `edgeTo[v]` is last edge on shortest path from s to v.

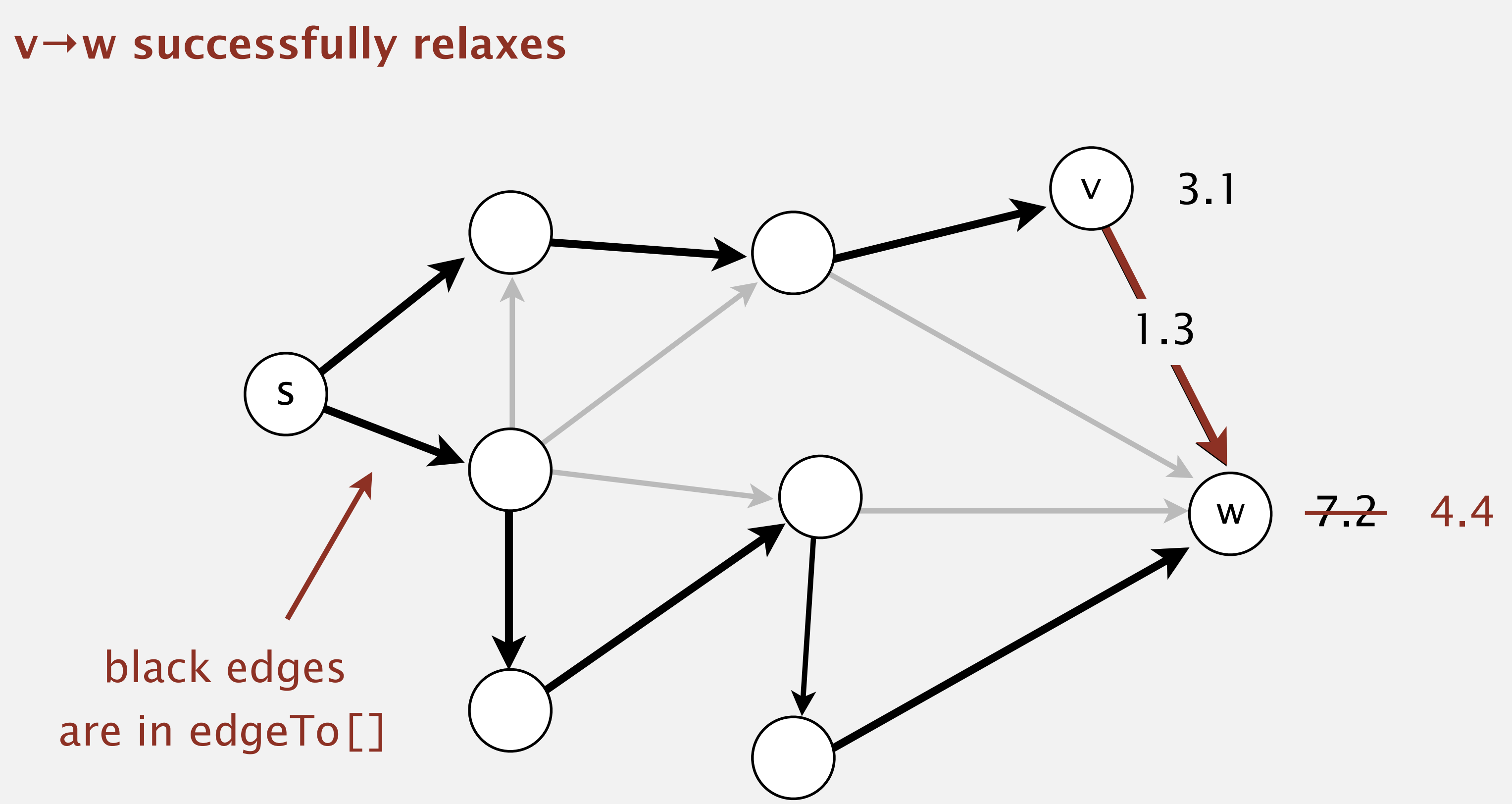
```
public double distTo(int v)
{  return distTo[v];  }

public Iterable<DirectedEdge> pathTo(int v)
{
   Stack<DirectedEdge> path = new Stack<DirectedEdge>();
   for (DirectedEdge e = edgeTo[v]; e != null; e = edgeTo[e.from()])
      path.push(e);
   return path;
}
```

Edge relaxation

Relax edge $e = v \rightarrow w$.

- `distTo[v]` is length of shortest known path from `s` to `v`.
- `distTo[w]` is length of shortest known path from `s` to `w`.
- `edgeTo[w]` is last edge on shortest known path from `s` to `w`.
- If `e = v→w` gives shorter path to `w` through `v`,
  update both `distTo[w]` and `edgeTo[w]`.



Edge relaxation

Relax edge $e = v \rightarrow w$.

- `distTo[v]` is length of shortest known path from `s` to `v`.
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- `edgeTo[w]` is last edge on shortest known path from `s` to `w`.
- If `e = v→w` gives shorter path to `w` through `v`,
  update both `distTo[w]` and `edgeTo[w]`.

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
   }
}
```

Edge relaxation

Relax edge $e = v \to w$.

- `distTo[v]` is length of shortest known path from `s` to `v`.
- `distTo[w]` is length of shortest known path from `s` to `w`.
- `edgeTo[w]` is last edge on shortest known path from `s` to `w`.
- If `e = v→w` gives shorter path to `w` through `v`,
  update both `distTo[w]` and `edgeTo[w]`.

```
private void relax(EdgeWeightedDigraph G, int v)
{
   for(DirectedEdge e: G.adj(V))
   {
      int w  = e.to();
      if (distTo[w] > distTo[v] + e.weight())
      {
          distTo[w] = distTo[v] + e.weight();
          edgeTo[w] = e;
      }
}
```

Shortest-paths optimality conditions

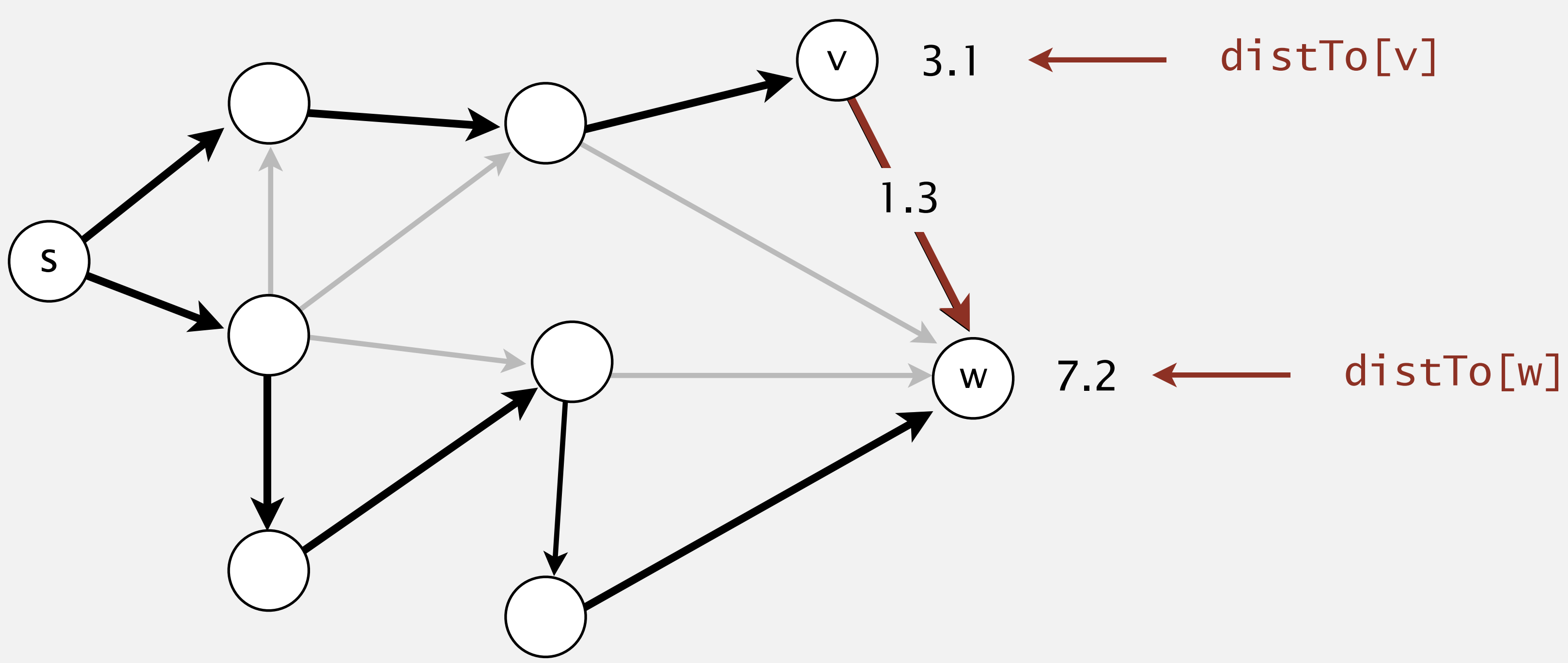
Proposition. Let G be an edge-weighted digraph.

Then `distTo[]` are the shortest path distances from `s` iff:

- `distTo[s]` = 0.
- For each vertex `v`, `distTo[v]` is the length of some path from `s` to `v`.
- For each edge `e` = `v→w`, `distTo[w] ≤ distTo[v] + e.weight()`.

Pf by contradiction. ⇐ [necessary]

- Suppose that `distTo[w] > distTo[v] + e.weight()` for some edge `e` = `v→w`.
- Then, `e` gives a path from `s` to `w` (through `v`) of length less than `distTo[w]`.



Shortest-paths optimality conditions

Proposition. Let G be an edge-weighted digraph.

Then `distTo[]` are the shortest path distances from `s` iff:

- `distTo[s]` $= 0$.
- For each vertex `v`, `distTo[v]` is the length of some path from `s` to `v`.
- For each edge `e = v→w`, `distTo[w] ≤ distTo[v] + e.weight()`.

Pf. $\Rightarrow$ [sufficient]

- Suppose that $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k = w$ is a shortest path from s to w.
- Then,

$$
\begin{aligned}
\texttt{distTo}[v_1] &\le \texttt{distTo}[v_0] + e_1\texttt{.weight()} \\
\texttt{distTo}[v_2] &\le \texttt{distTo}[v_1] + e_2\texttt{.weight()} \\
&\ldots \\
\texttt{distTo}[v_k] &\le \texttt{distTo}[v_{k-1}] + e_k\texttt{.weight()}
\end{aligned}
$$

e_i = i^{th} edge on shortest path from s to w

- Add inequalities; simplify; and substitute $\texttt{distTo}[v_0] = \texttt{distTo[s]} = 0$:

$$\texttt{distTo[w]} = \texttt{distTo}[v_k] \le \underbrace{e_1\texttt{.weight()} + e_2\texttt{.weight()} + \ldots + e_k\texttt{.weight()}}_{\text{weight of shortest path from s to w}}$$

- Thus, `distTo[w]` is the weight of shortest path to w. ▪

weight of some path from s to w

Generic shortest-paths algorithm

Generic algorithm (to compute SPT from s)

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat until optimality conditions are satisfied:

- **Relax any edge.**

Proposition. Generic algorithm computes SPT (if it exists) from `s`.

Pf sketch.

- The entry `distTo[v]` is always the length of a simple path from `s` to `v`.
- Each successful relaxation decreases `distTo[v]` for some `v`.
- The entry `distTo[v]` can decrease at most a finite number of times. ▪

Generic shortest-paths algorithm

Generic algorithm (to compute SPT from s)

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat until optimality conditions are satisfied:

- **Relax any edge.**

Efficient implementations. How to choose which edge to relax?

Ex 1. Dijkstra's algorithm (nonnegative weights).

Ex 2. Topological sort algorithm (no directed cycles).

Ex 3. Bellman-Ford algorithm (no negative cycles).

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4.4 Shortest Paths

- APIs
- shortest-paths properties
- Dijkstra's algorithm
- edge-weighted DAGs
- negative weights

Edsger W. Dijkstra: select quotes

" Do only what only you can do. "

" In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind. "

" The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence. "

" It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration. "

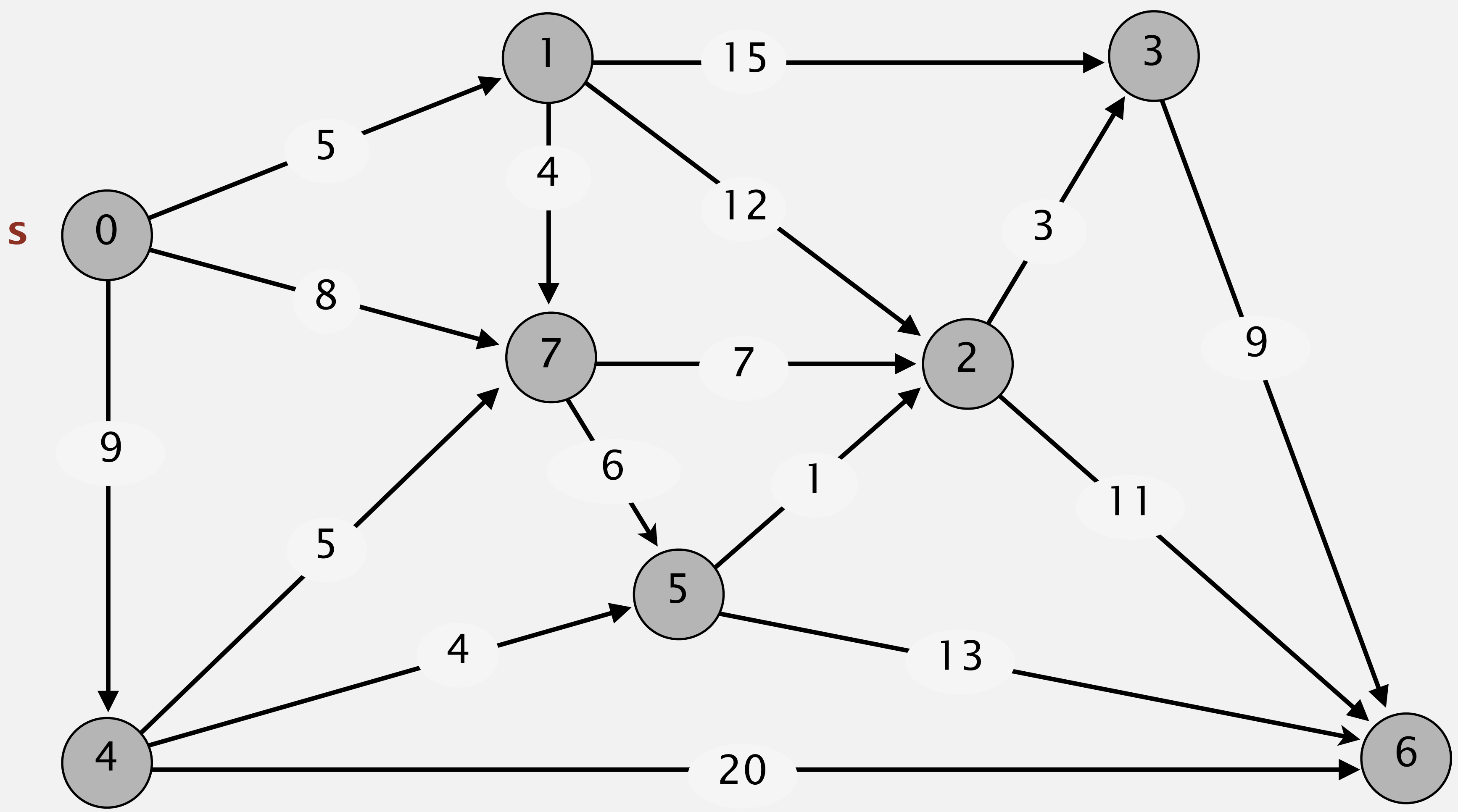
" APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums. "

Edsger W. Dijkstra
Turing award 1972

Edsger W. Dijkstra: select quotes

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s`
  (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.



an edge-weighted digraph

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

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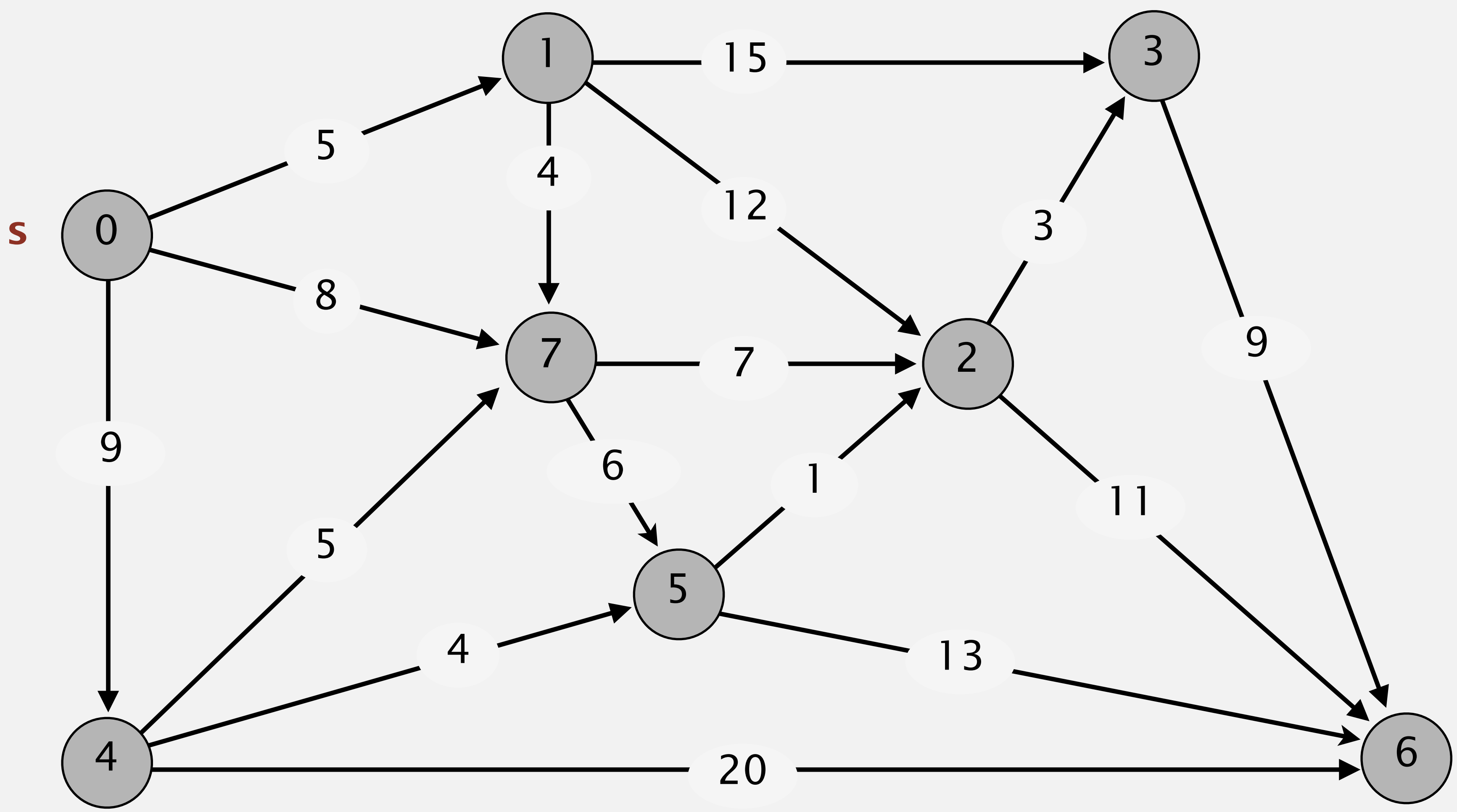

an edge-weighted digraph

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
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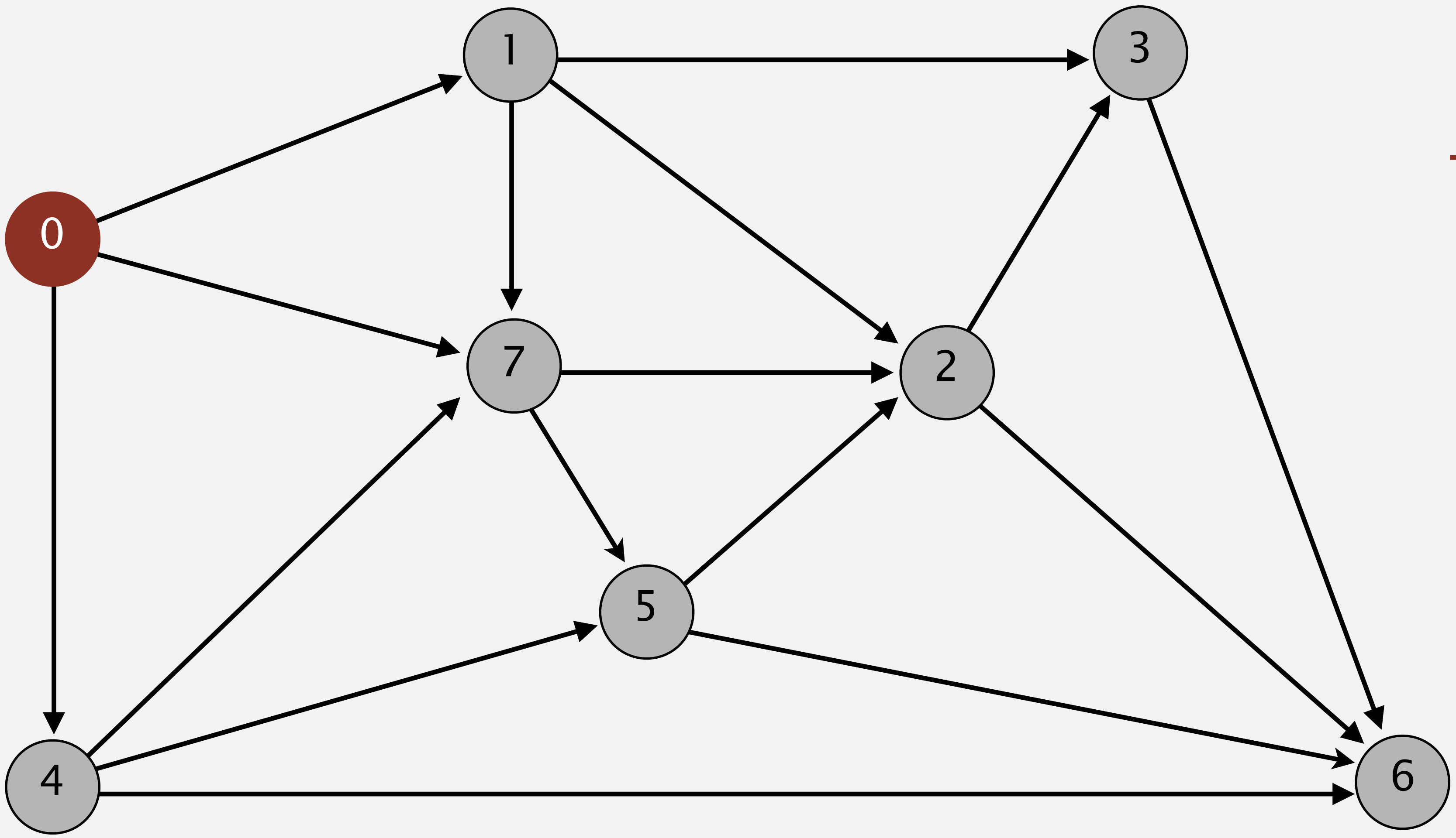
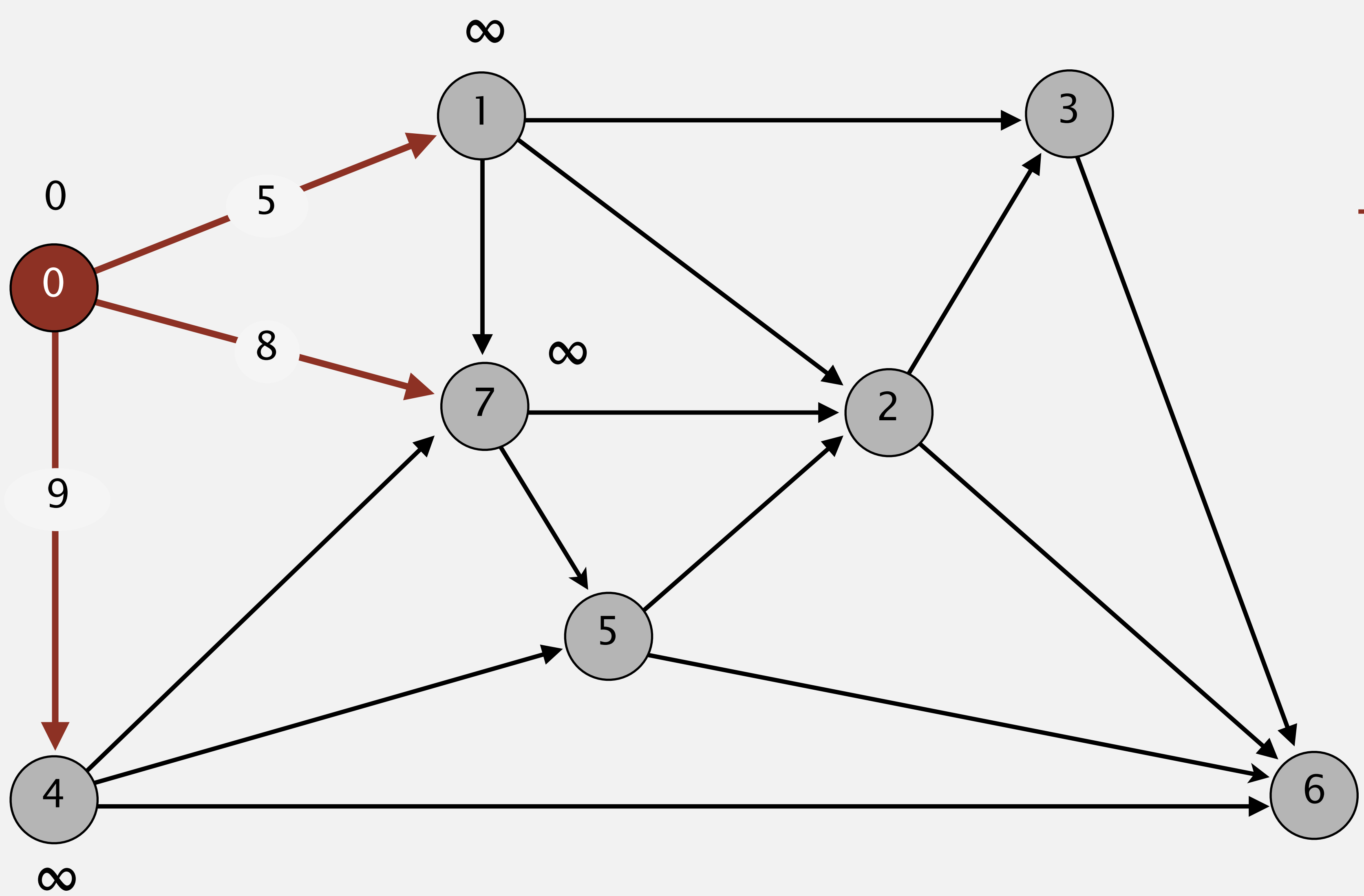


v	distTo[]	edgeTo[]
0	0.0	-
1		
2		
3		
4		
5		
6		
7		

choose source vertex 0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s`
  (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

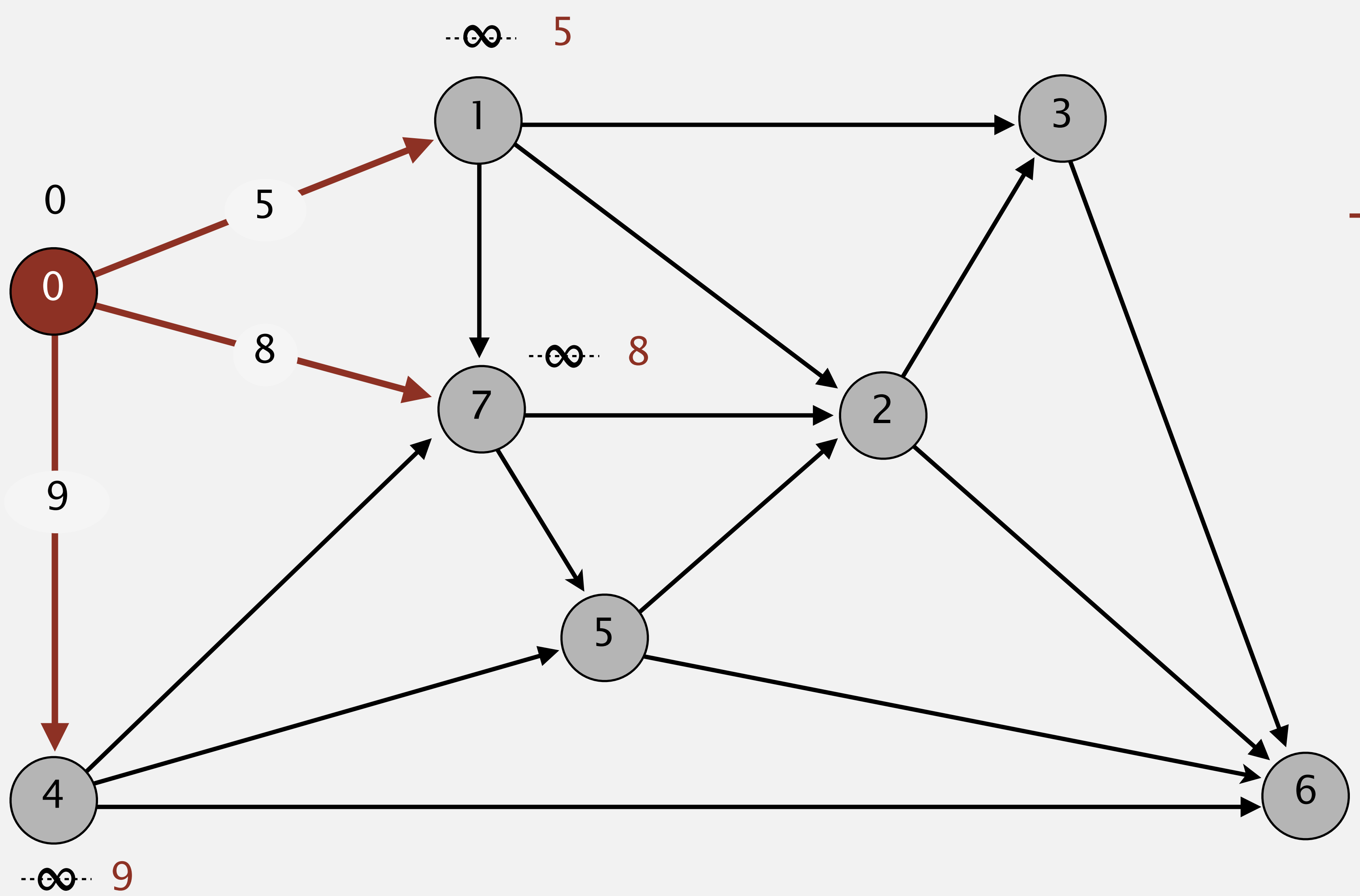


v	distTo[]	edgeTo[]
0	0.0	-
1		
2		
3		
4		
5		
6		
7		

relax all edges pointing from 0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

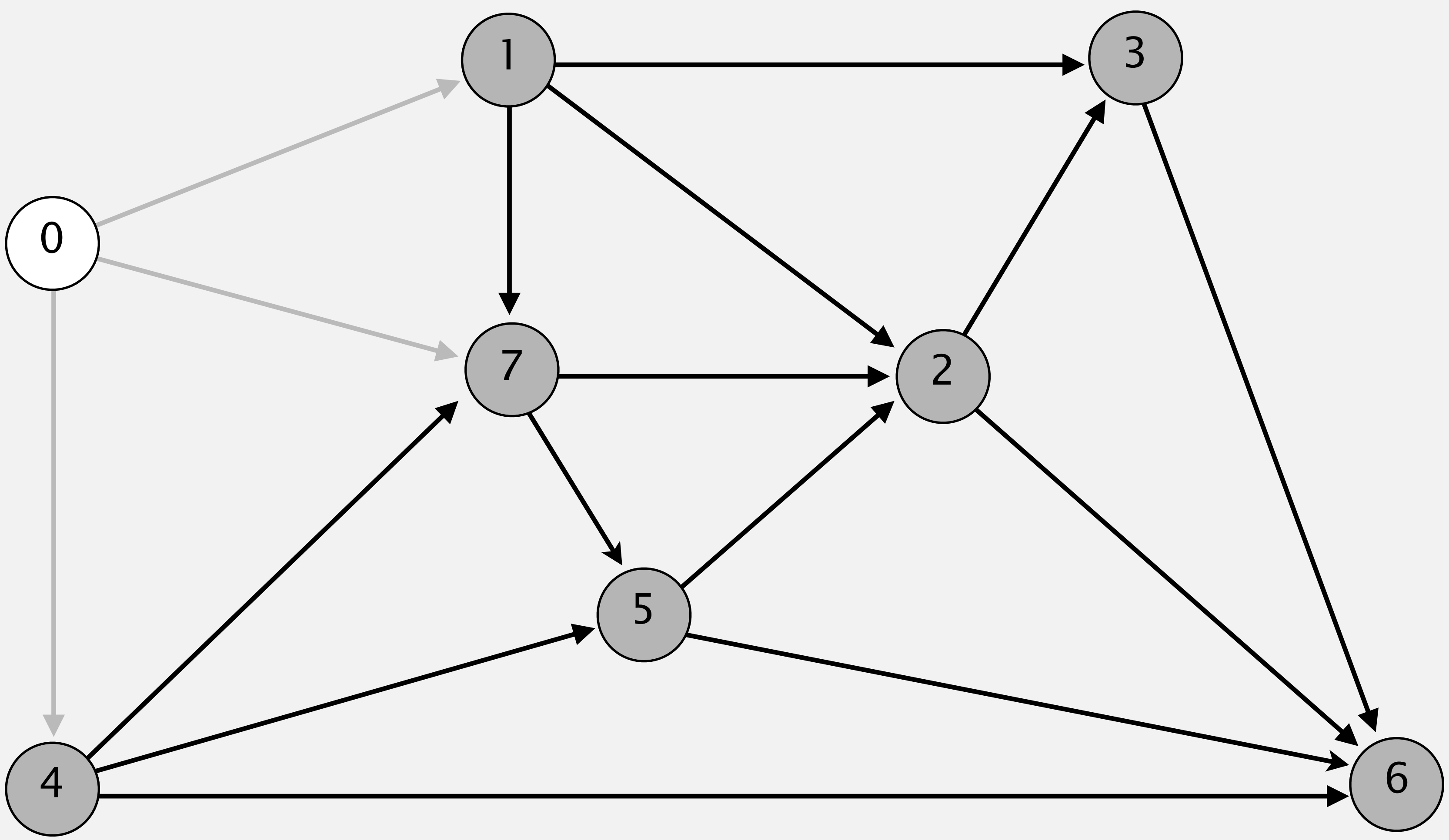


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

relax all edges pointing from 0

Dijkstra's algorithm demo

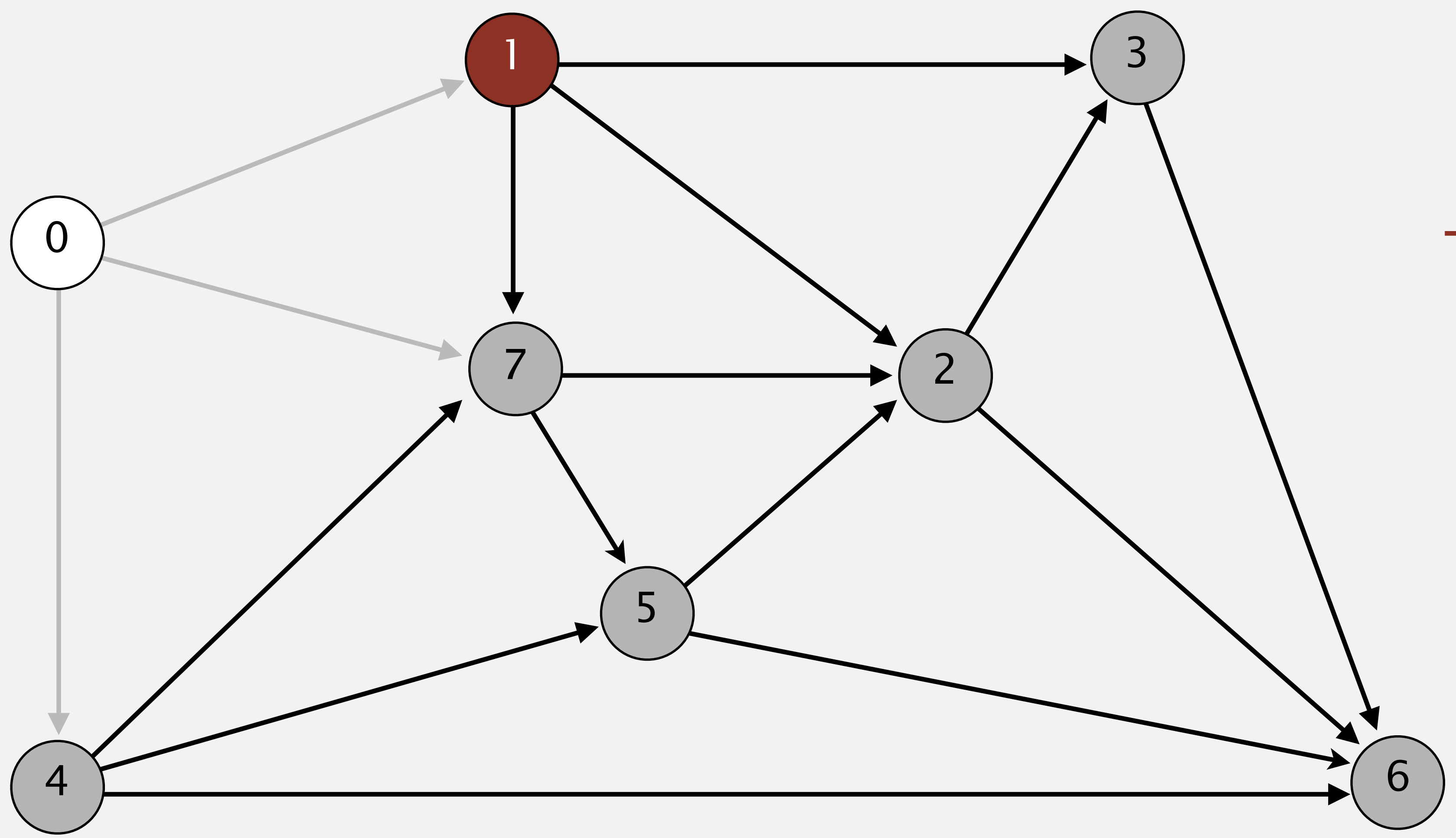
- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

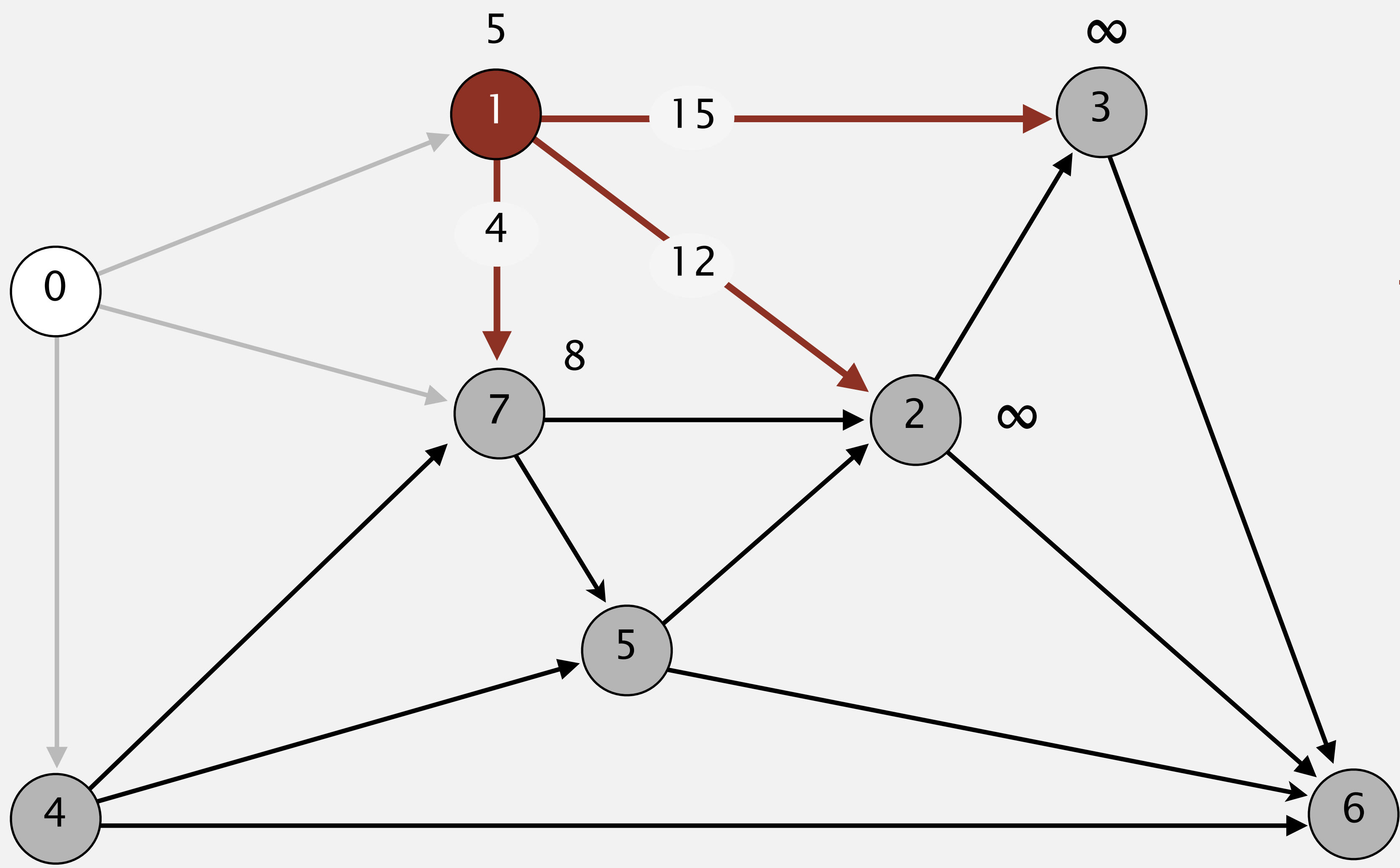


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

choose vertex 1

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

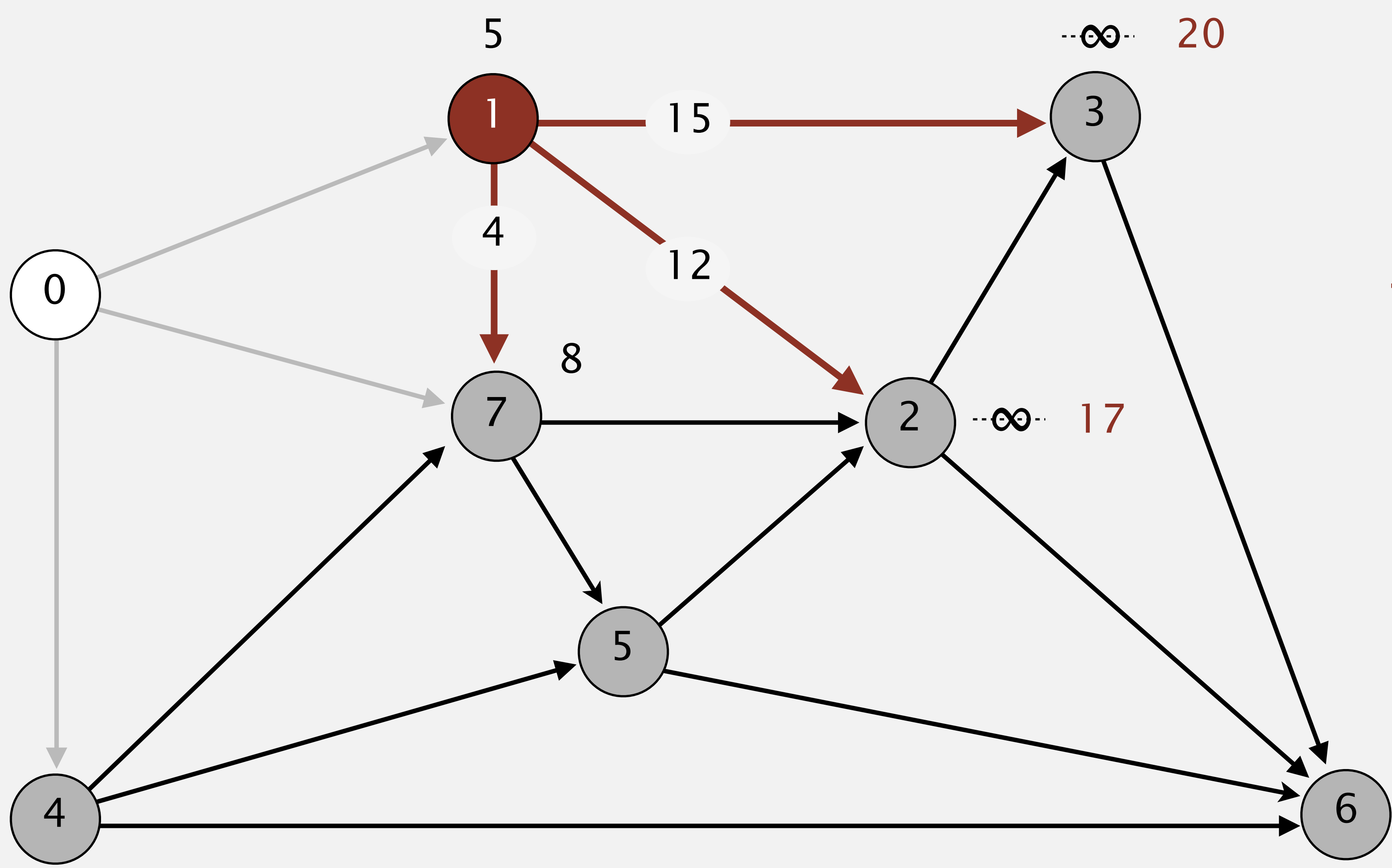


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

relax all edges pointing from 1

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

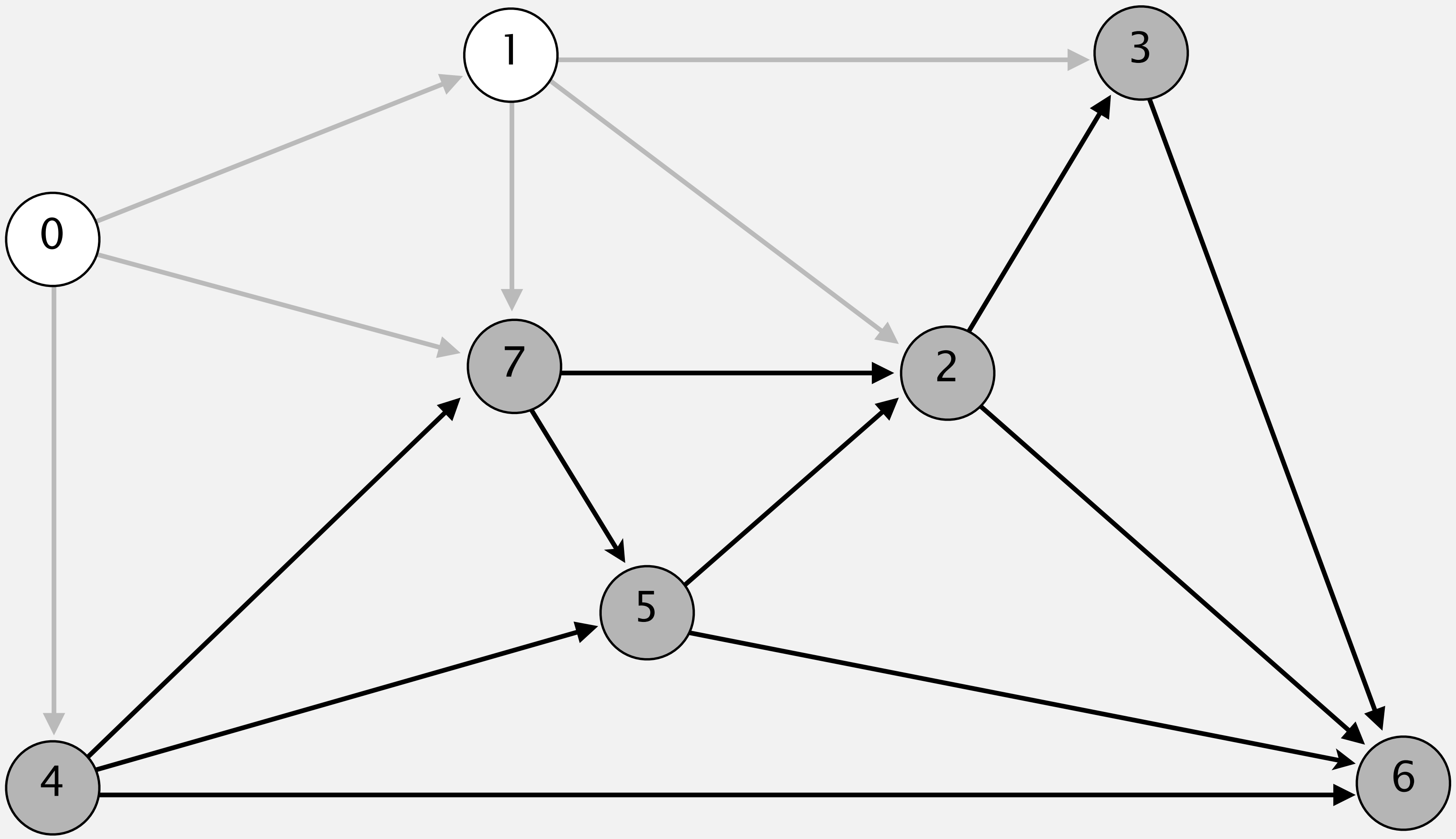


v	distTo[]	edgeTo[]
0	0.0	-
→ 1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0 ✔	0→7

relax all edges pointing from 1

Dijkstra's algorithm demo

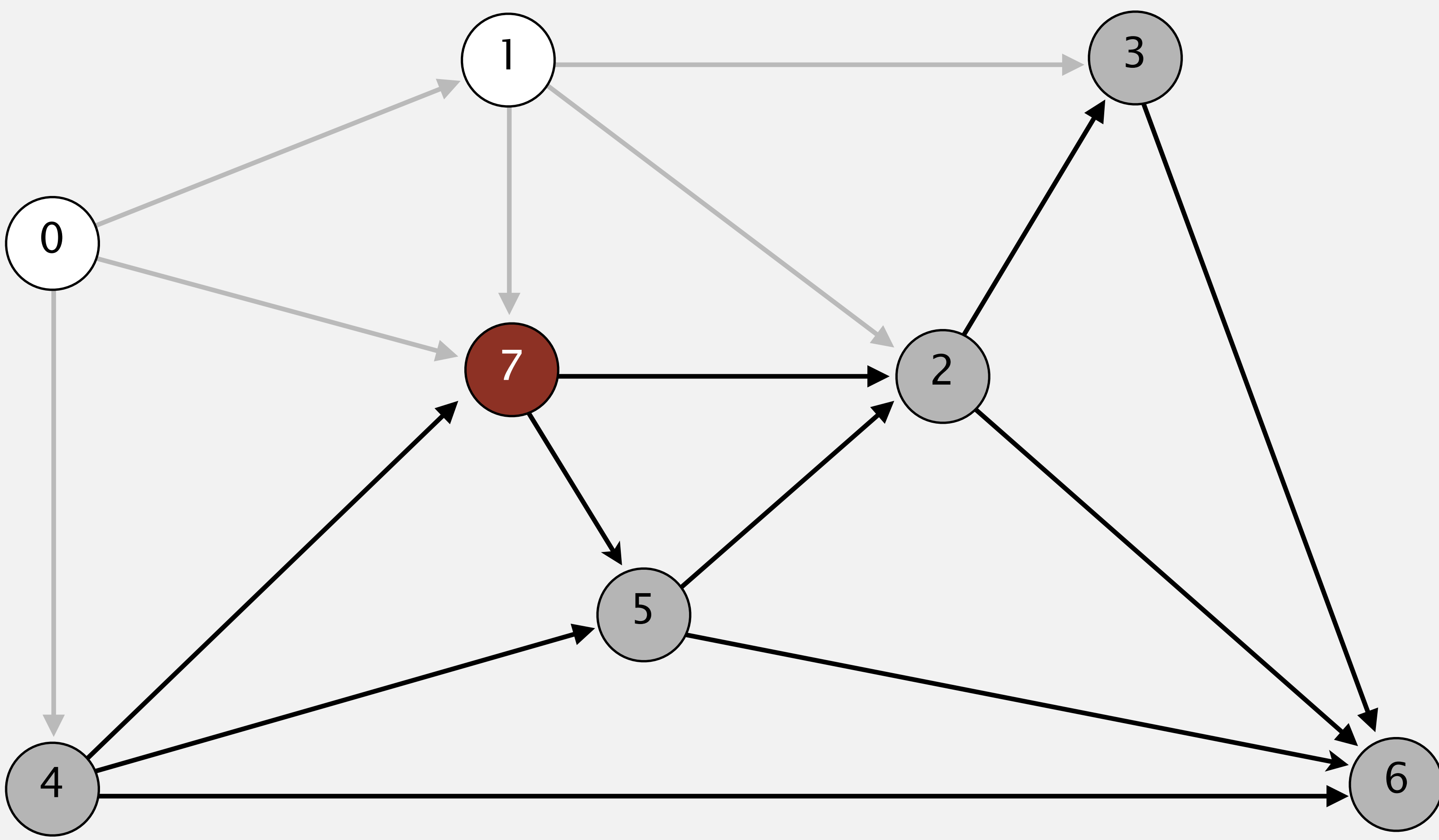
- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

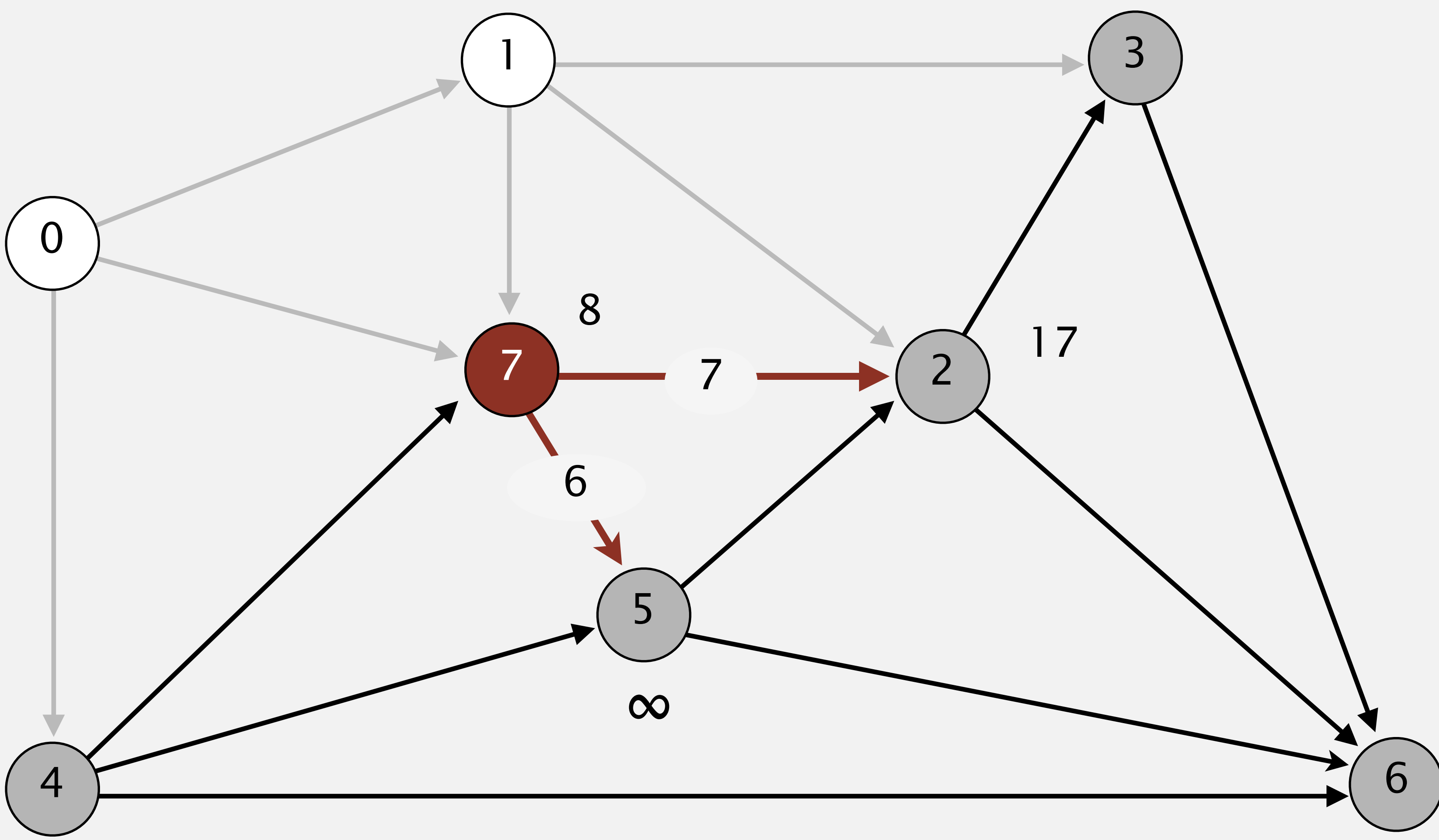


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

choose vertex 7

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

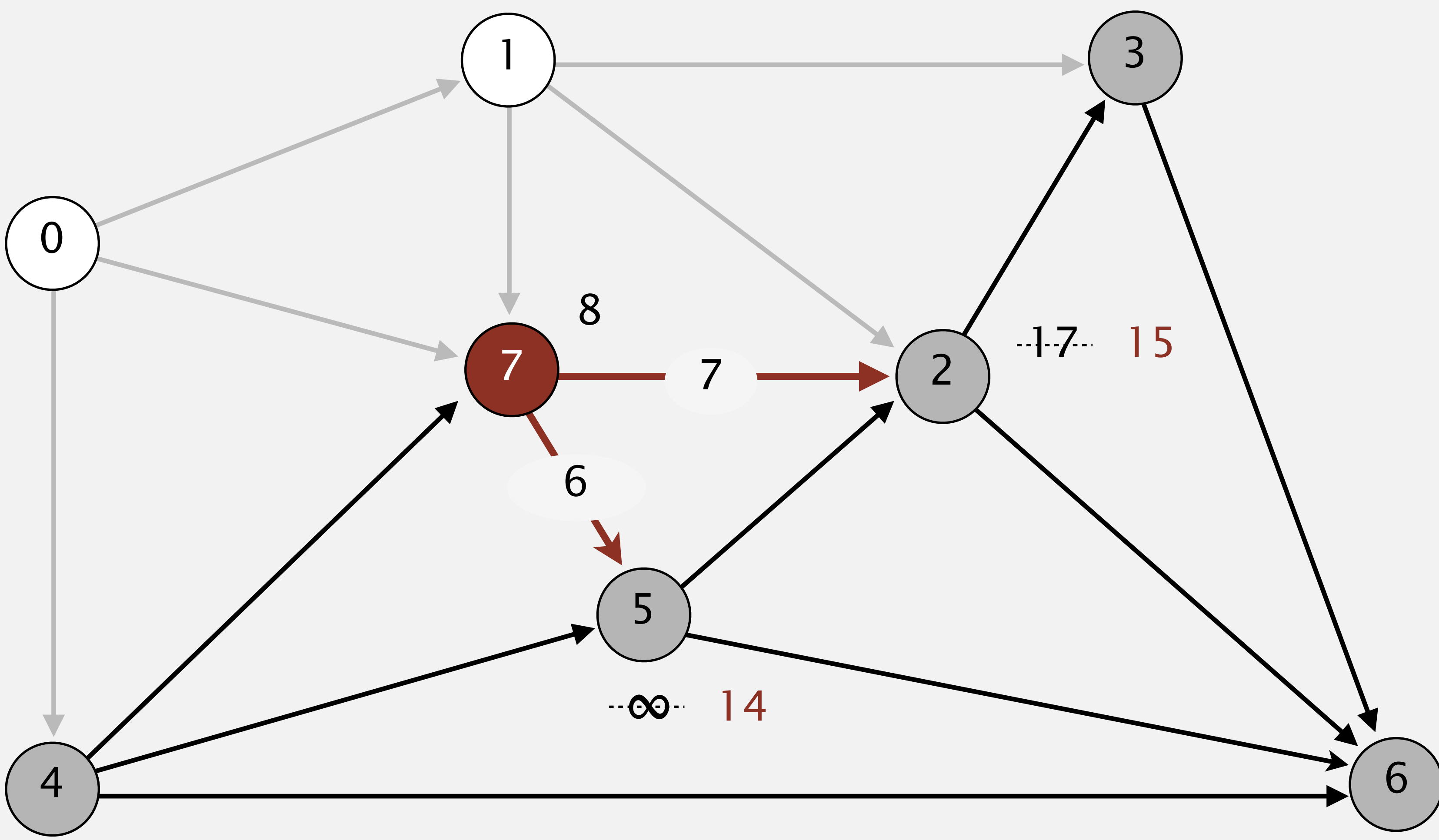


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
→ 7	8.0	0→7

relax all edges pointing from 7

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

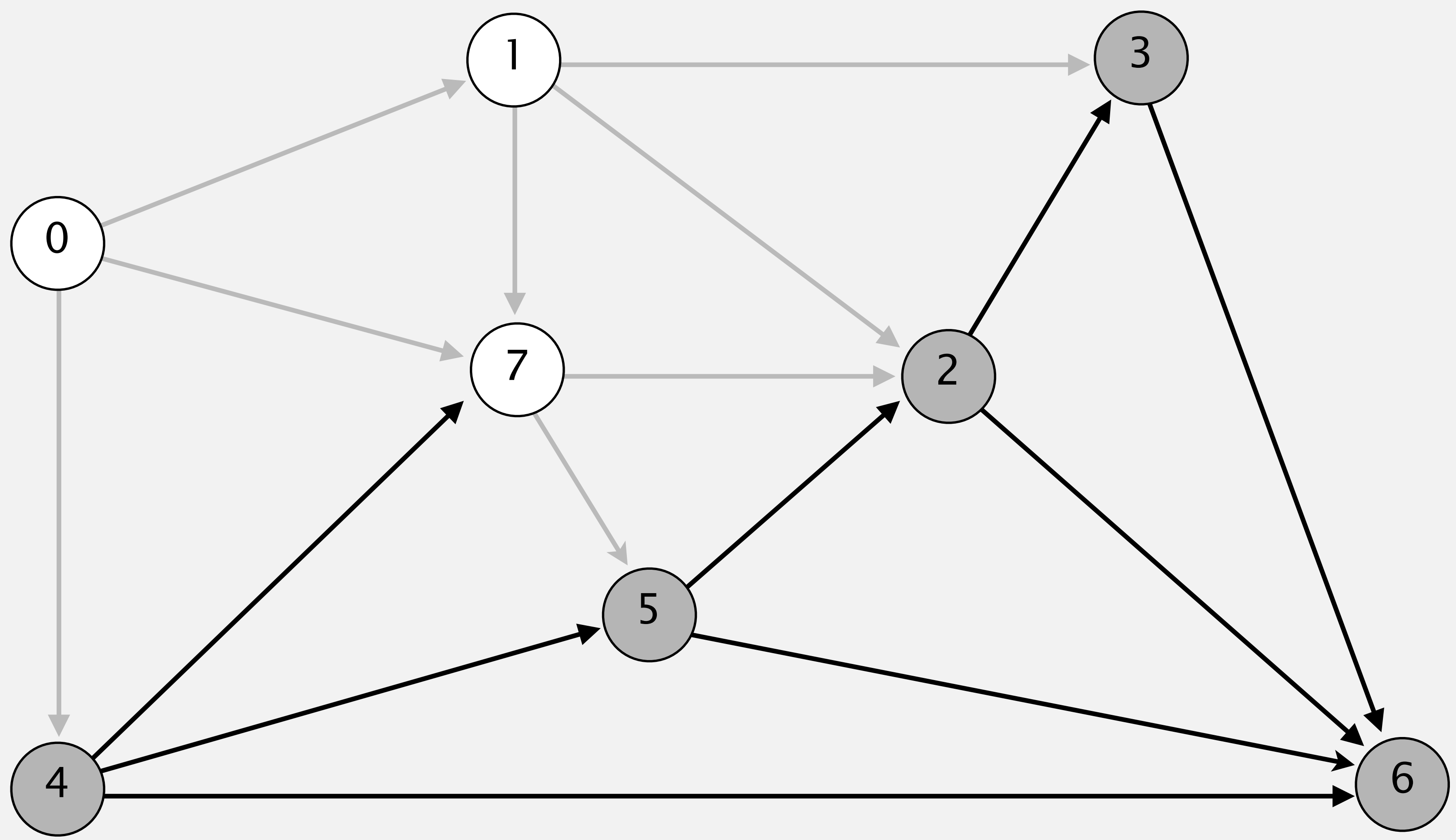


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	14.0	7→5
6		
7	8.0	0→7

relax all edges pointing from 7

Dijkstra's algorithm demo

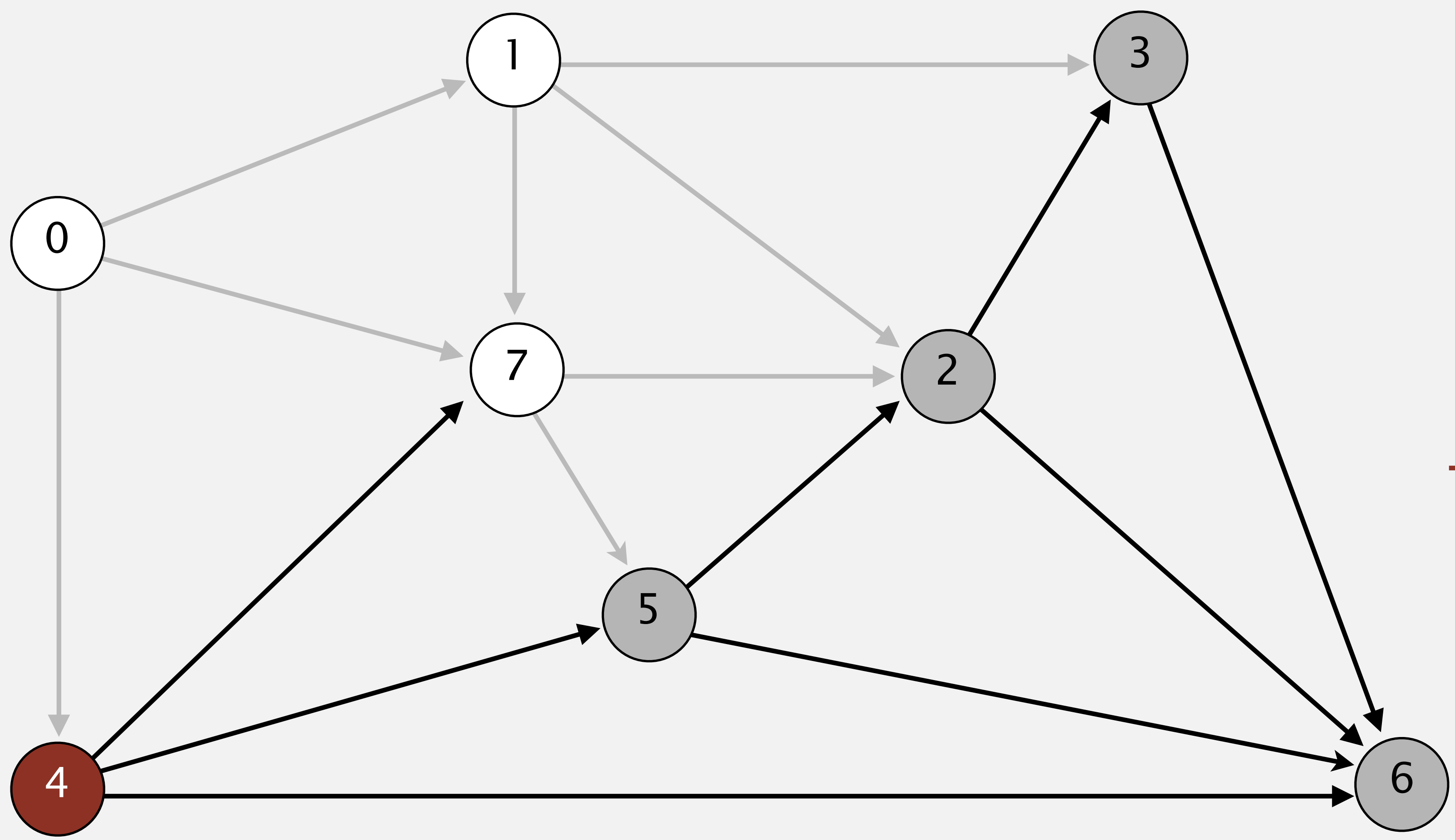
- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	14.0	7→5
6		
7	8.0	0→7

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

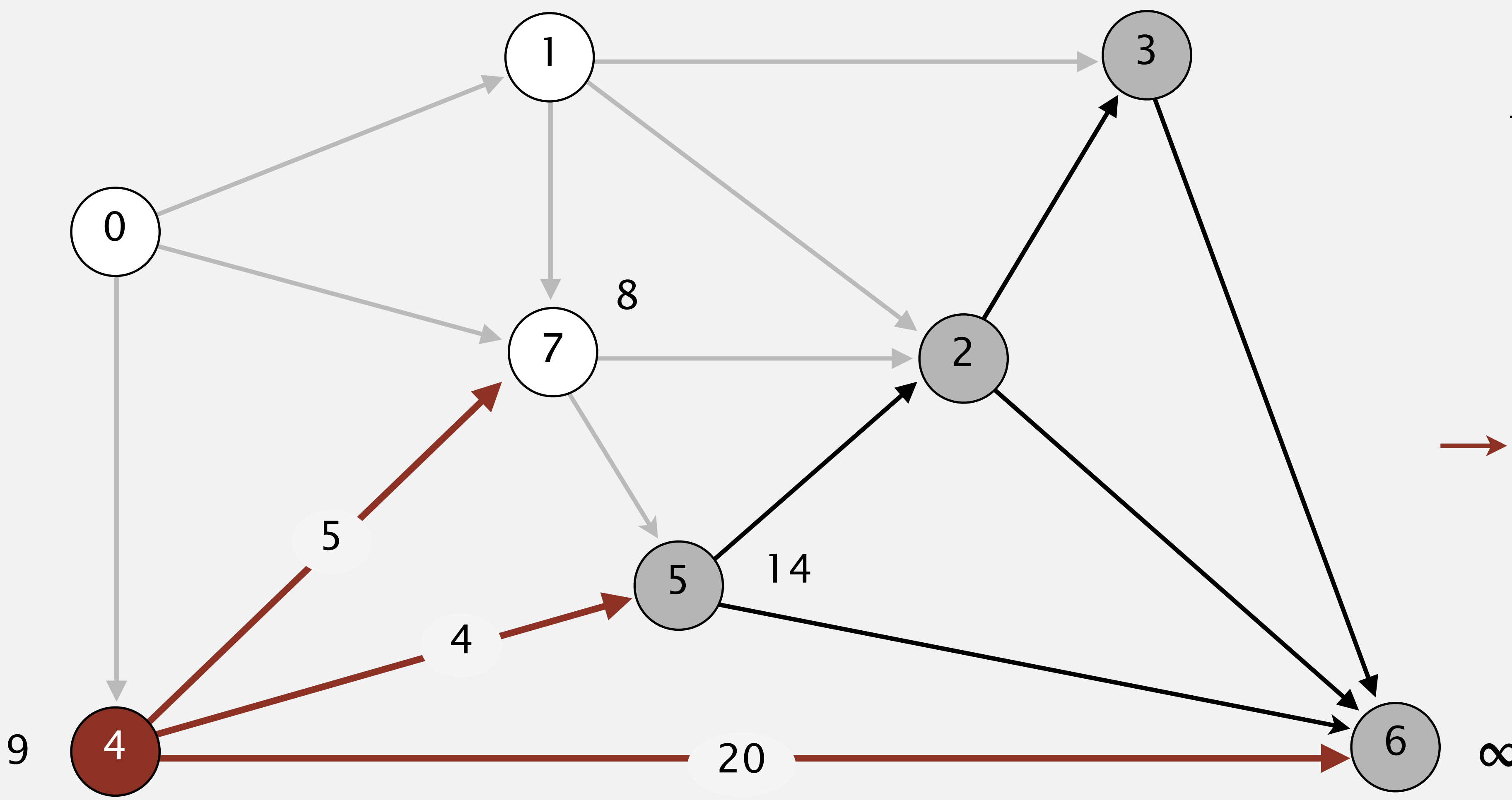


	v	distTo[]	edgeTo[]
	0	0.0	-
	1	5.0	0→1
	2	15.0	7→2
	3	20.0	1→3
→	4	9.0	0→4
	5	14.0	7→5
	6		
	7	8.0	0→7

select vertex 4

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

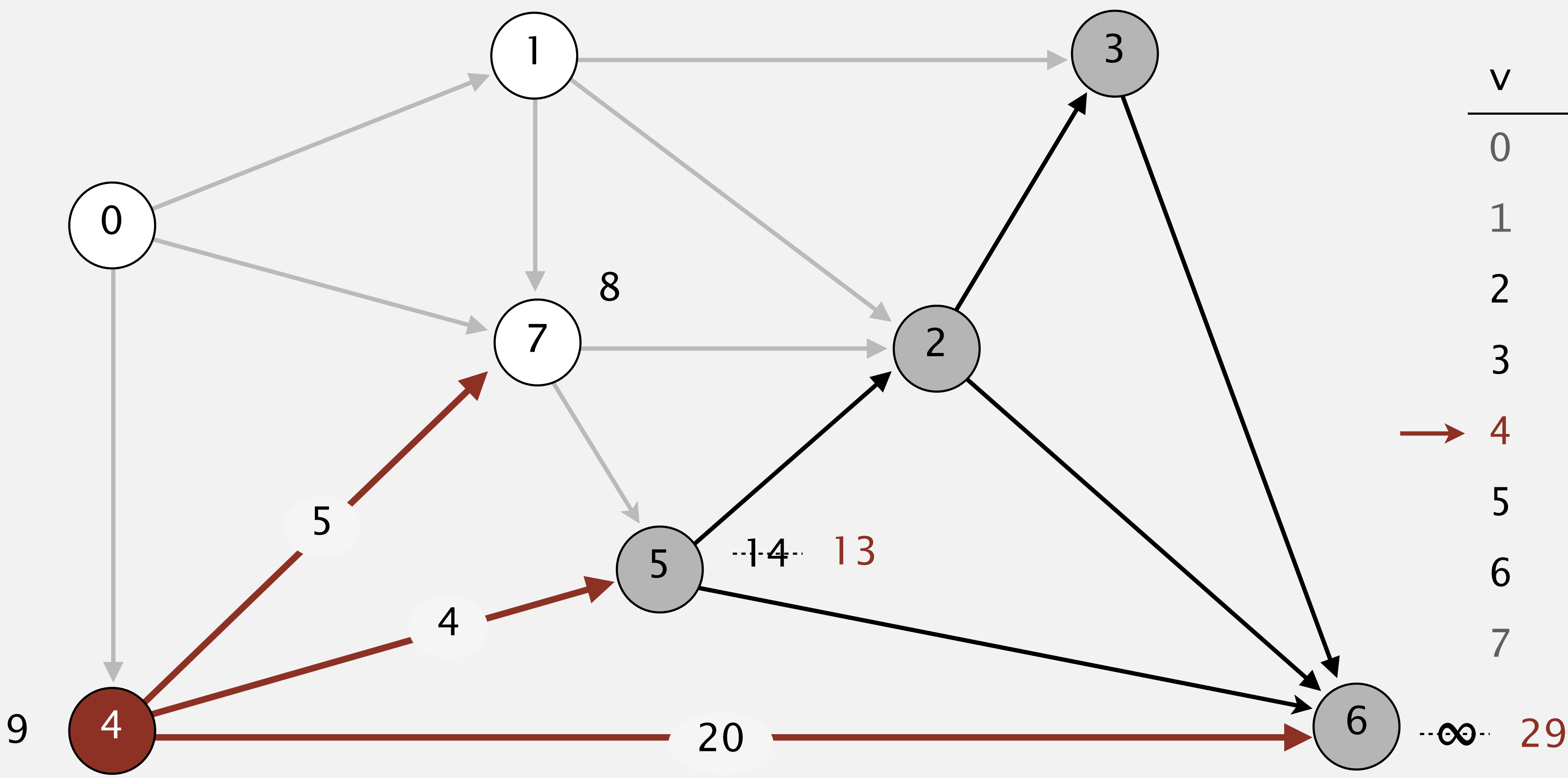


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	14.0	7→5
6		
7	8.0	0→7

relax all edges pointing from 4

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s`
  (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

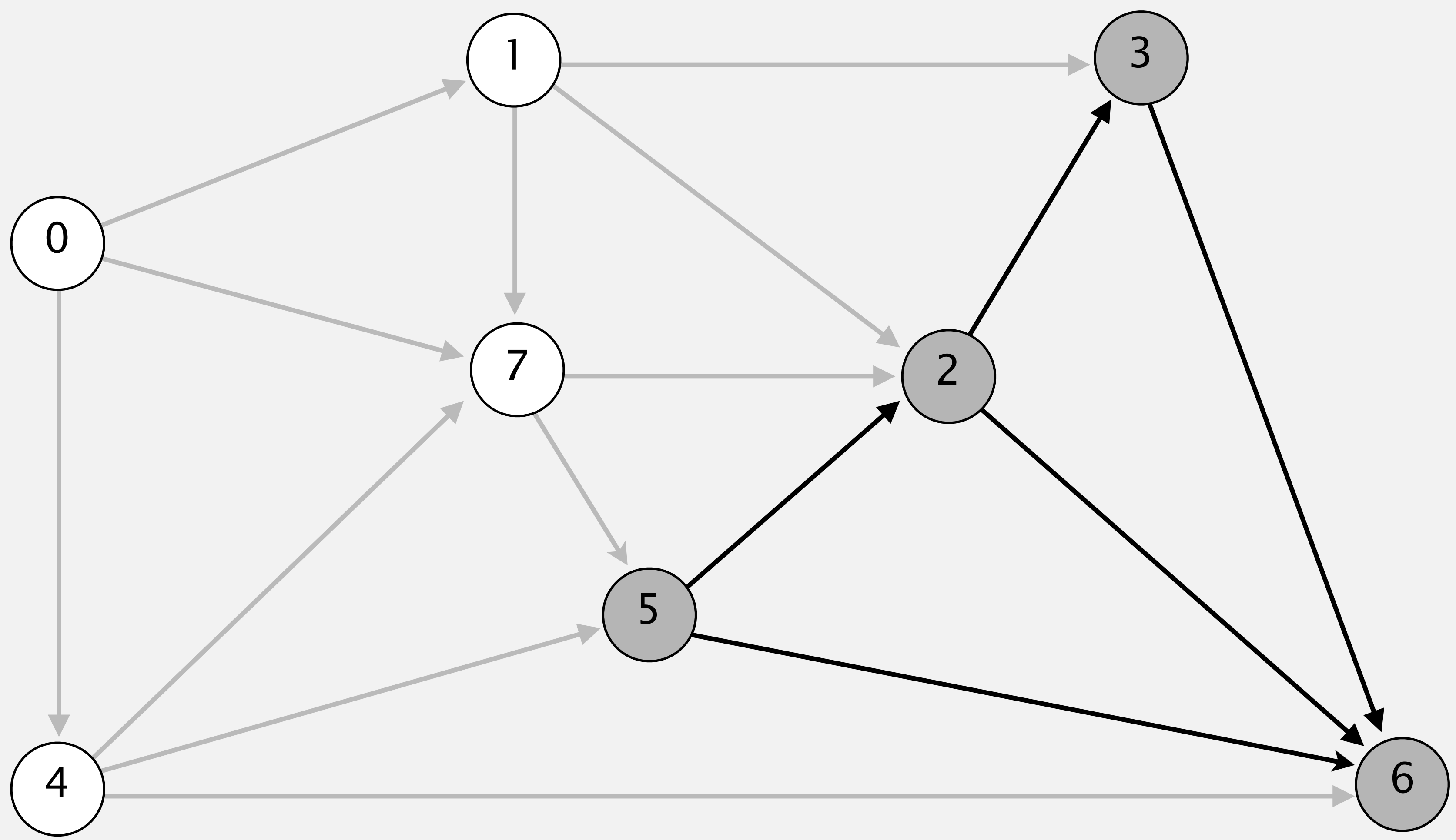


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	29.0	4→6
7	8.0 ✔	0→7

relax all edges pointing from 4

Dijkstra's algorithm demo

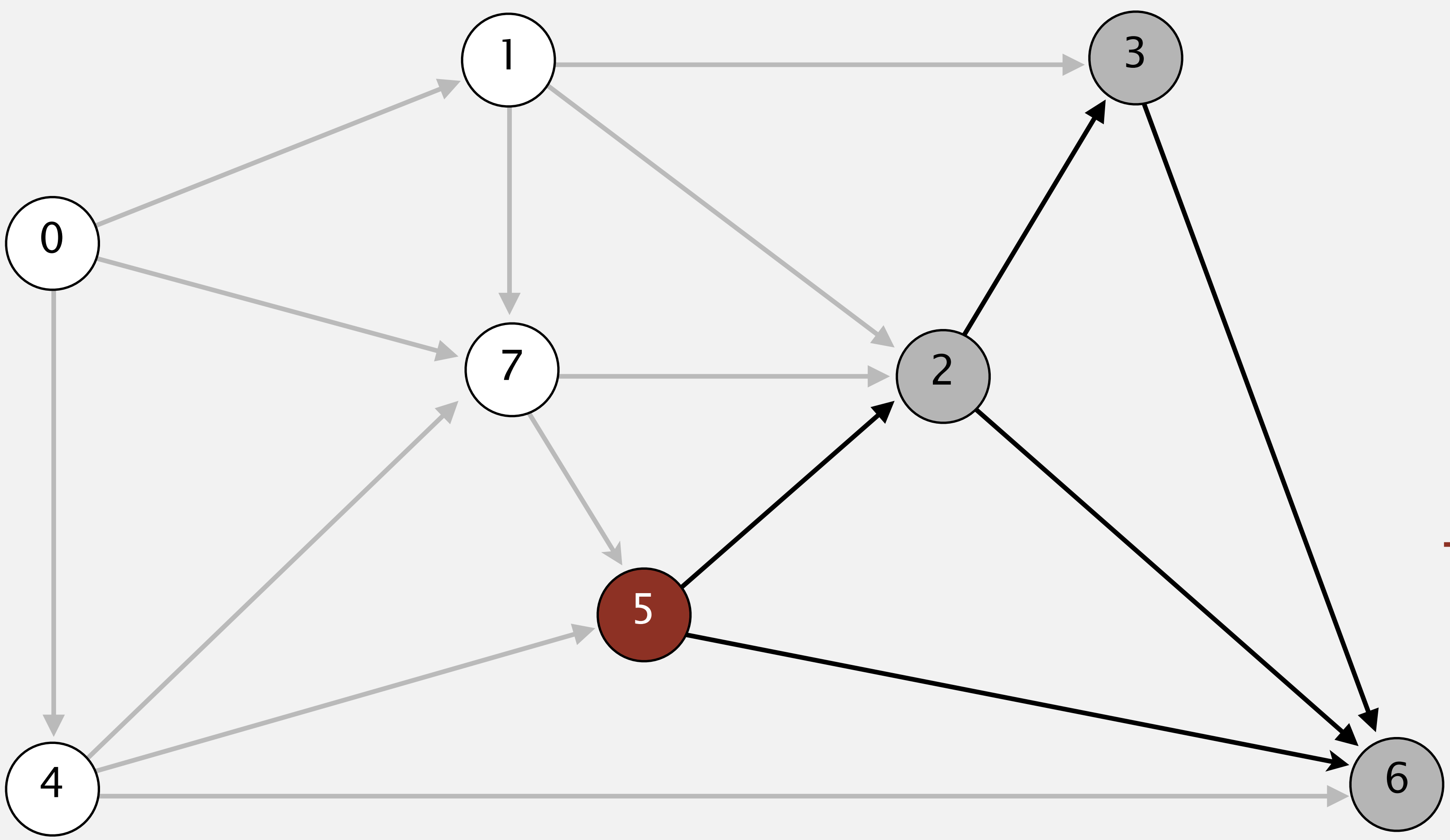
- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	29.0	4→6
7	8.0	0→7

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

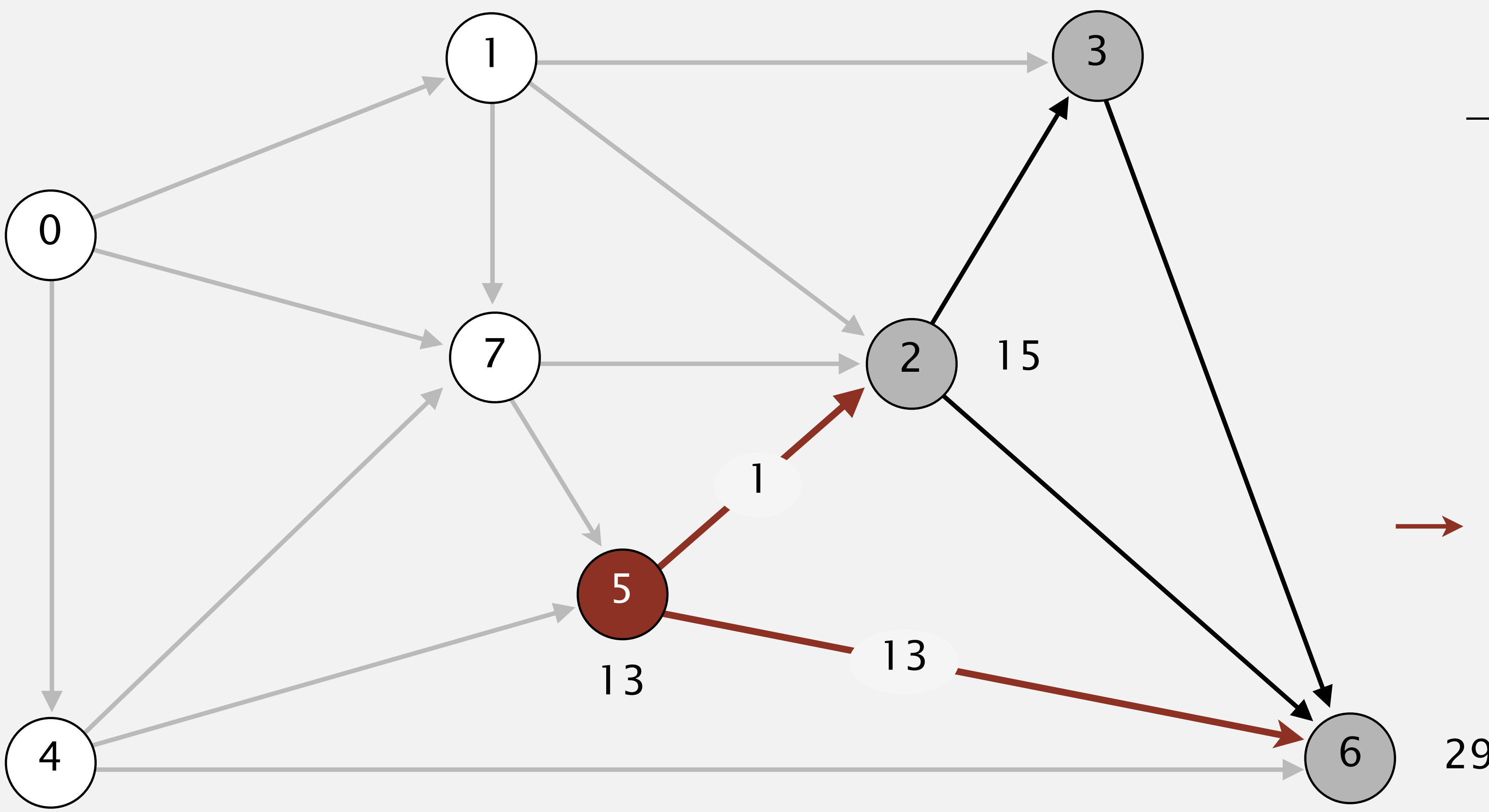


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
→ 5	13.0	4→5
6	29.0	4→6
7	8.0	0→7

select vertex 5

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

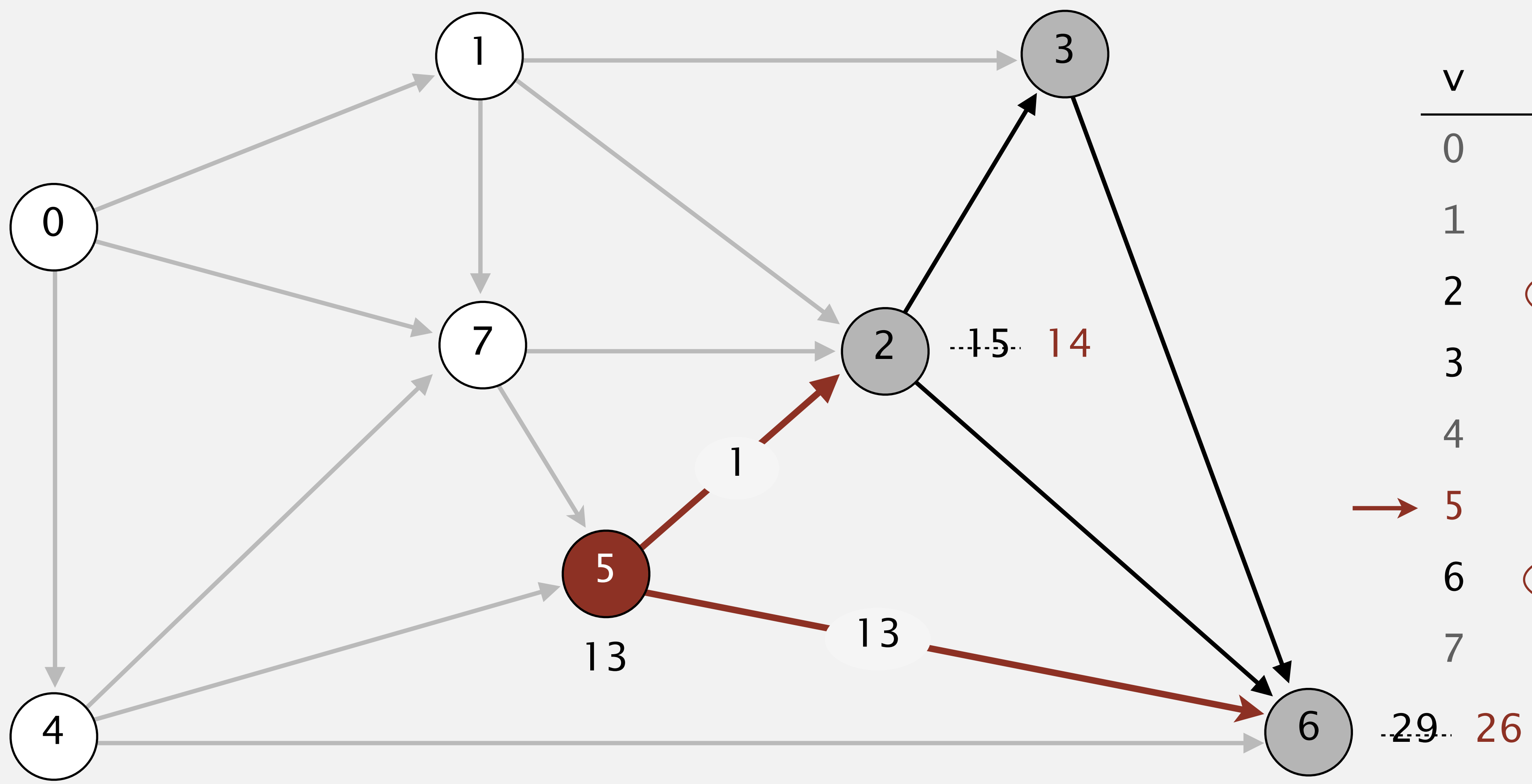


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
→ 5	13.0	4→5
6	29.0	4→6
7	8.0	0→7

relax all edges pointing from 5

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

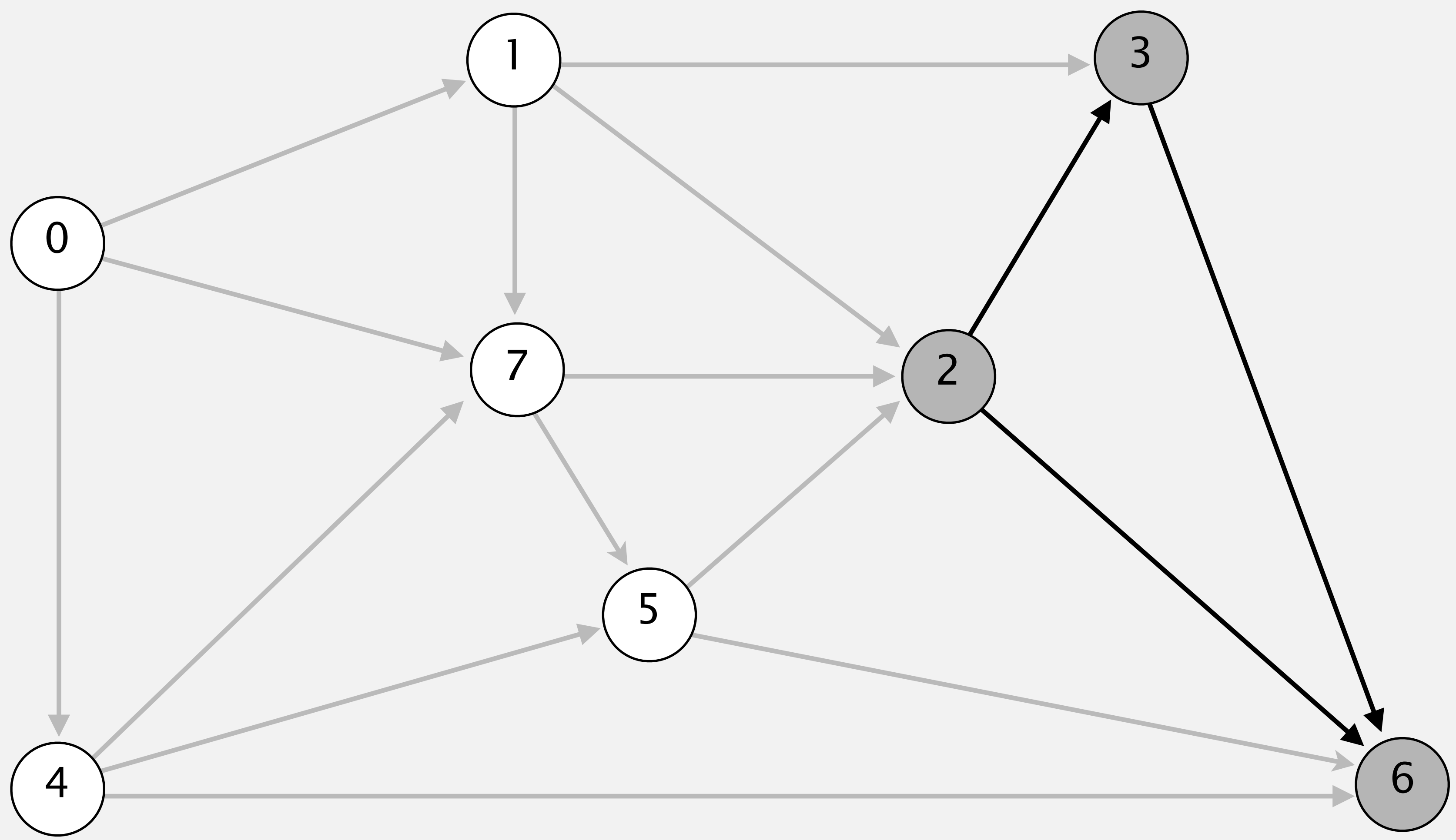


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

relax all edges pointing from 5

Dijkstra's algorithm demo

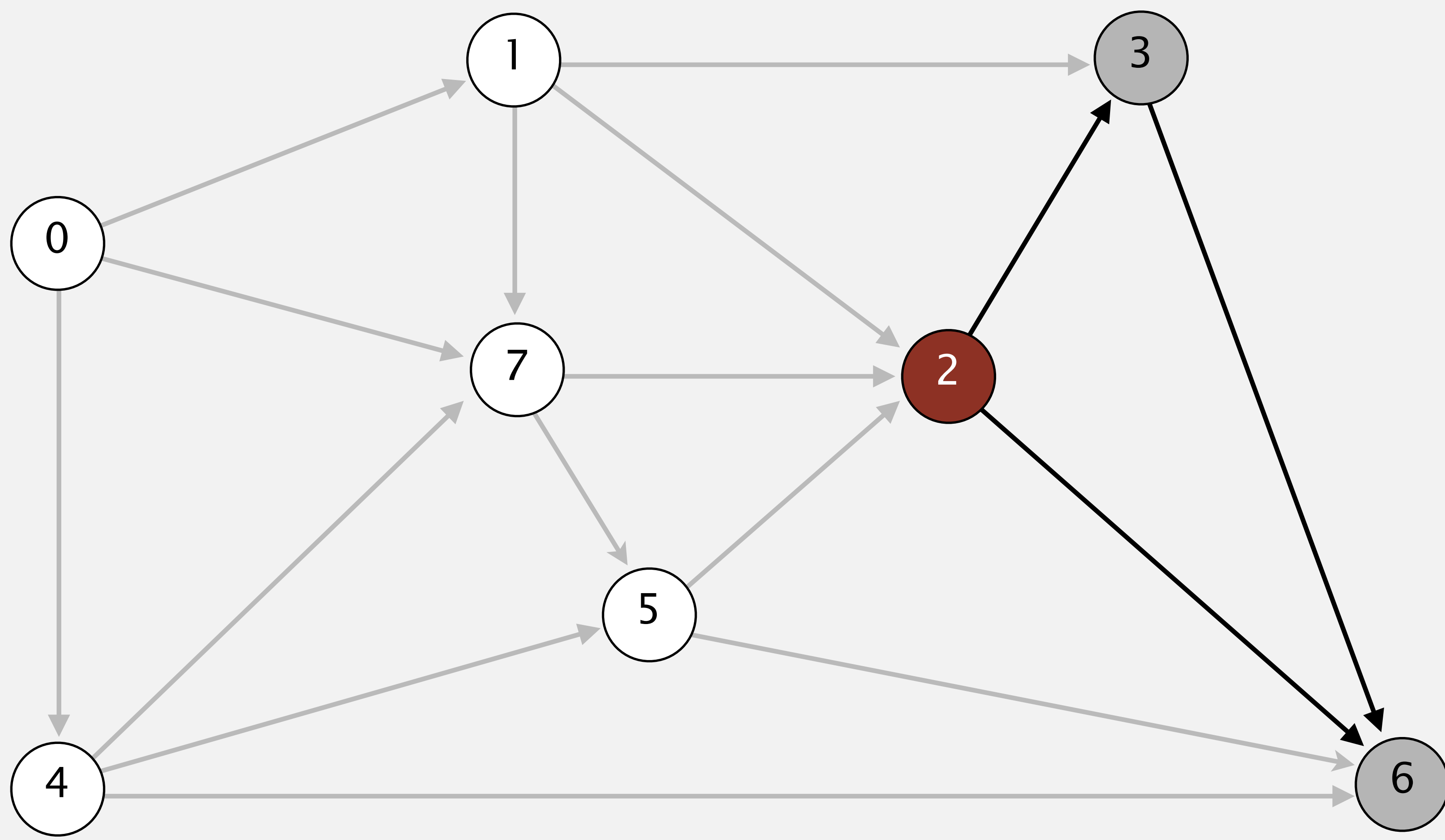
- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

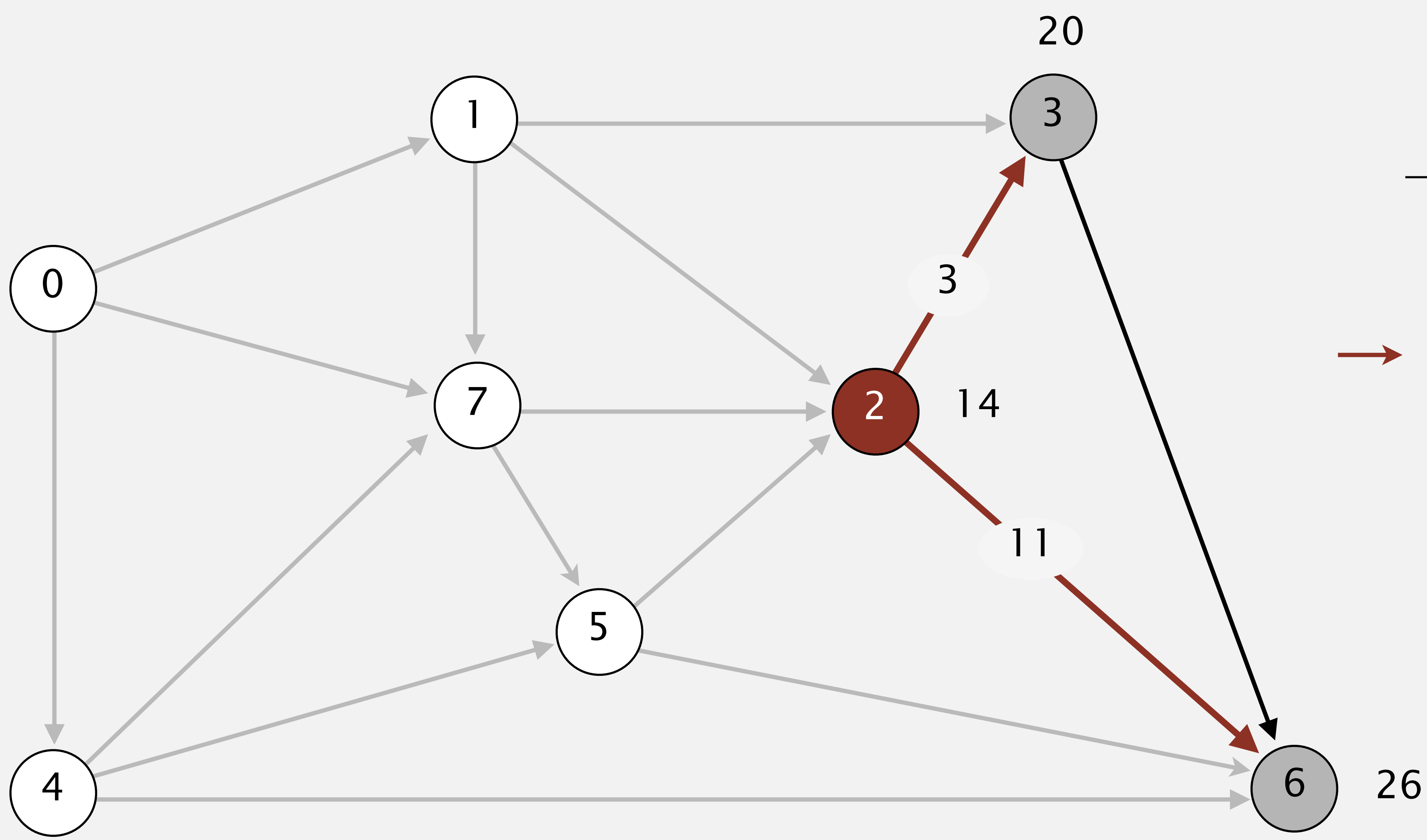


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

select vertex 2

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

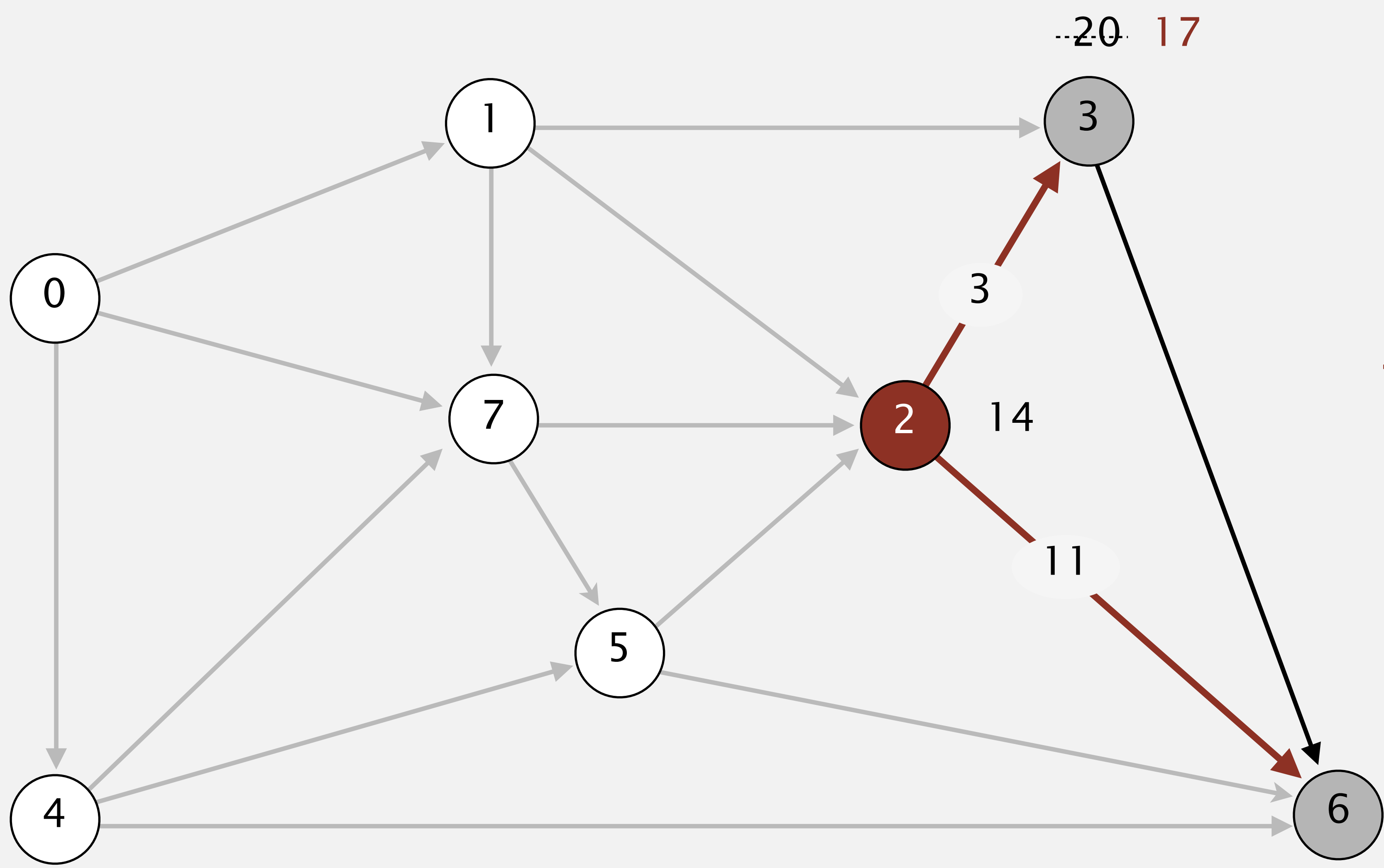


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

relax all edges pointing from 2

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges pointing from that vertex.

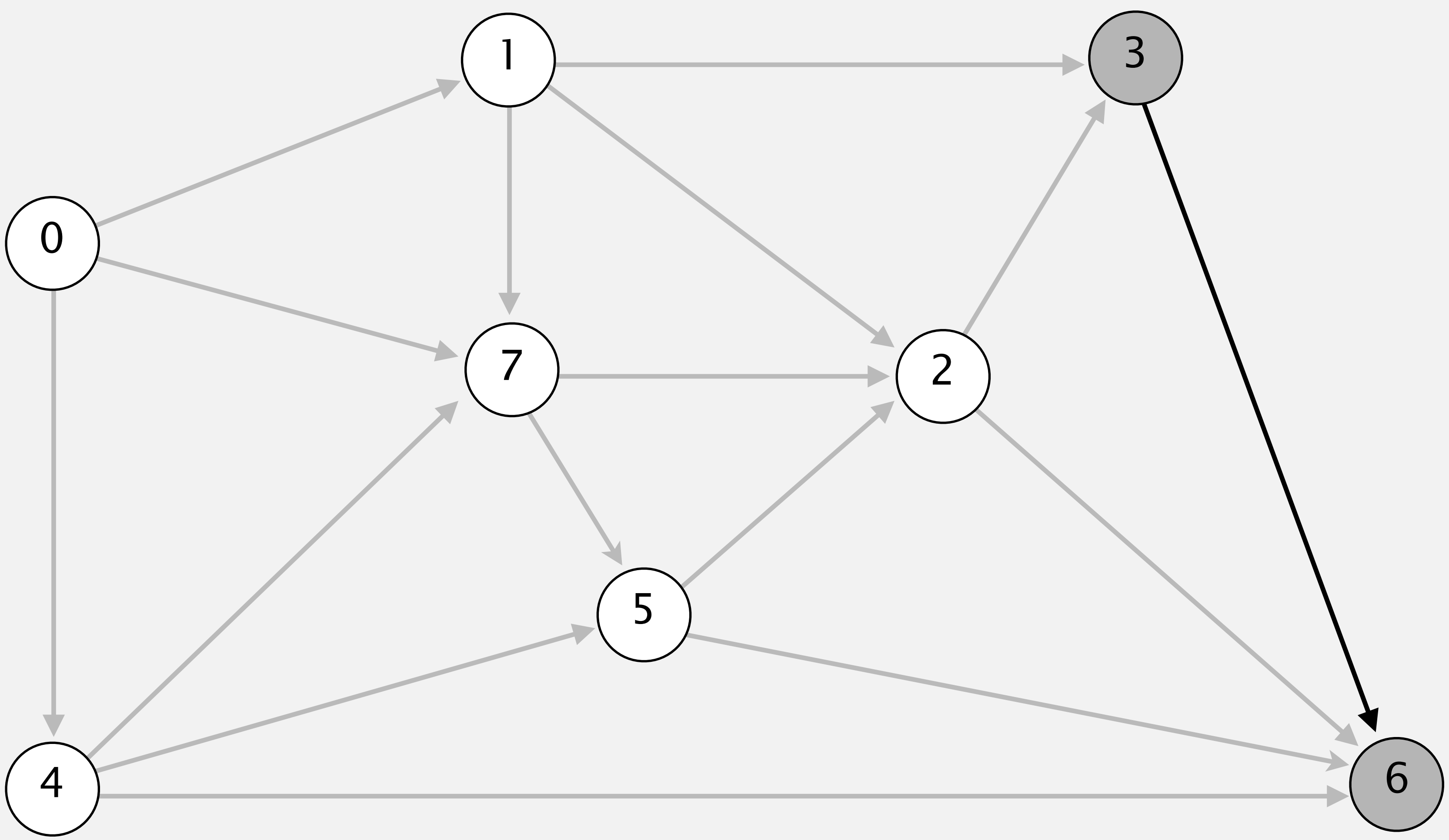


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

relax all edges pointing from 2

Dijkstra's algorithm demo

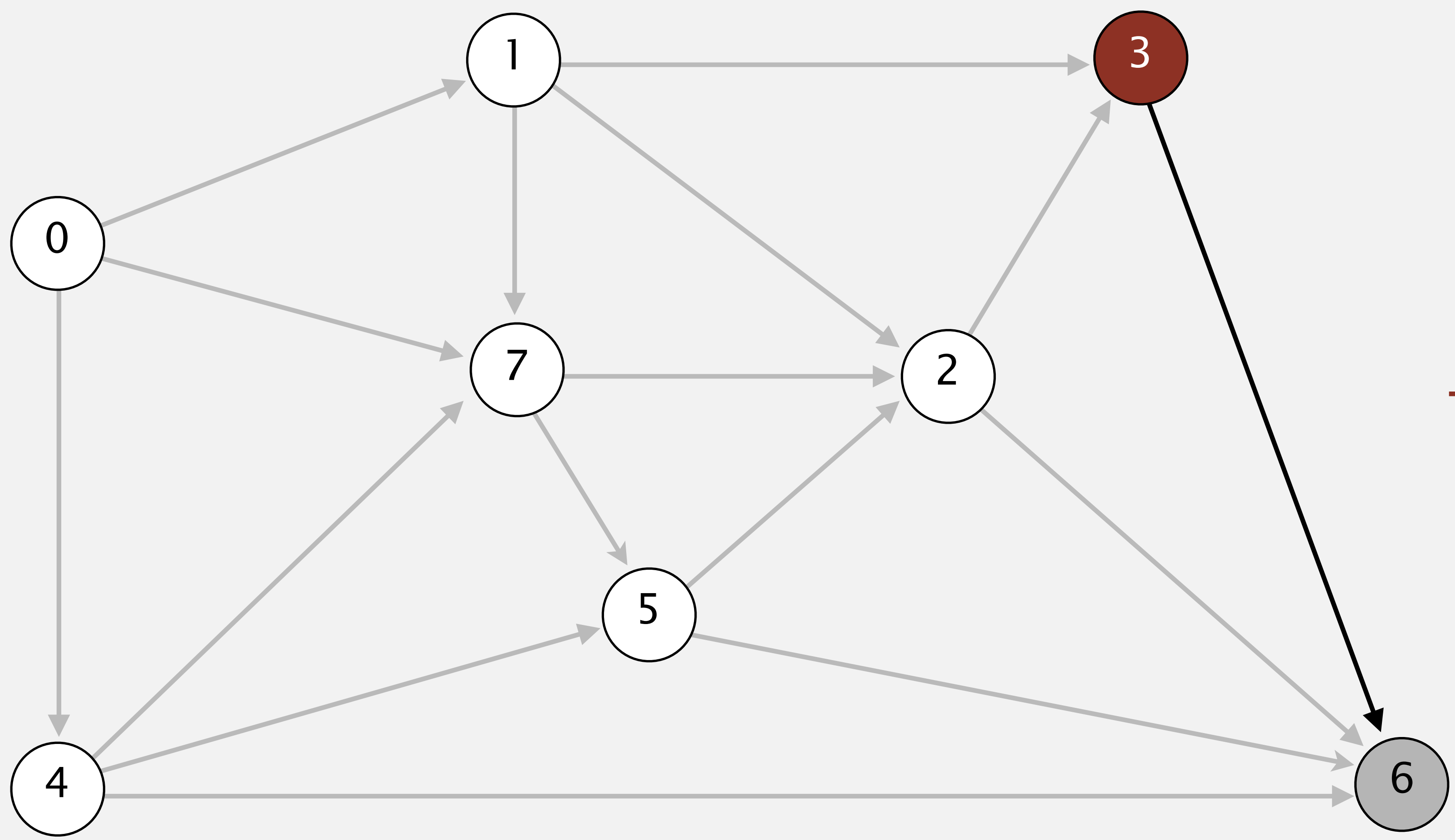
- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

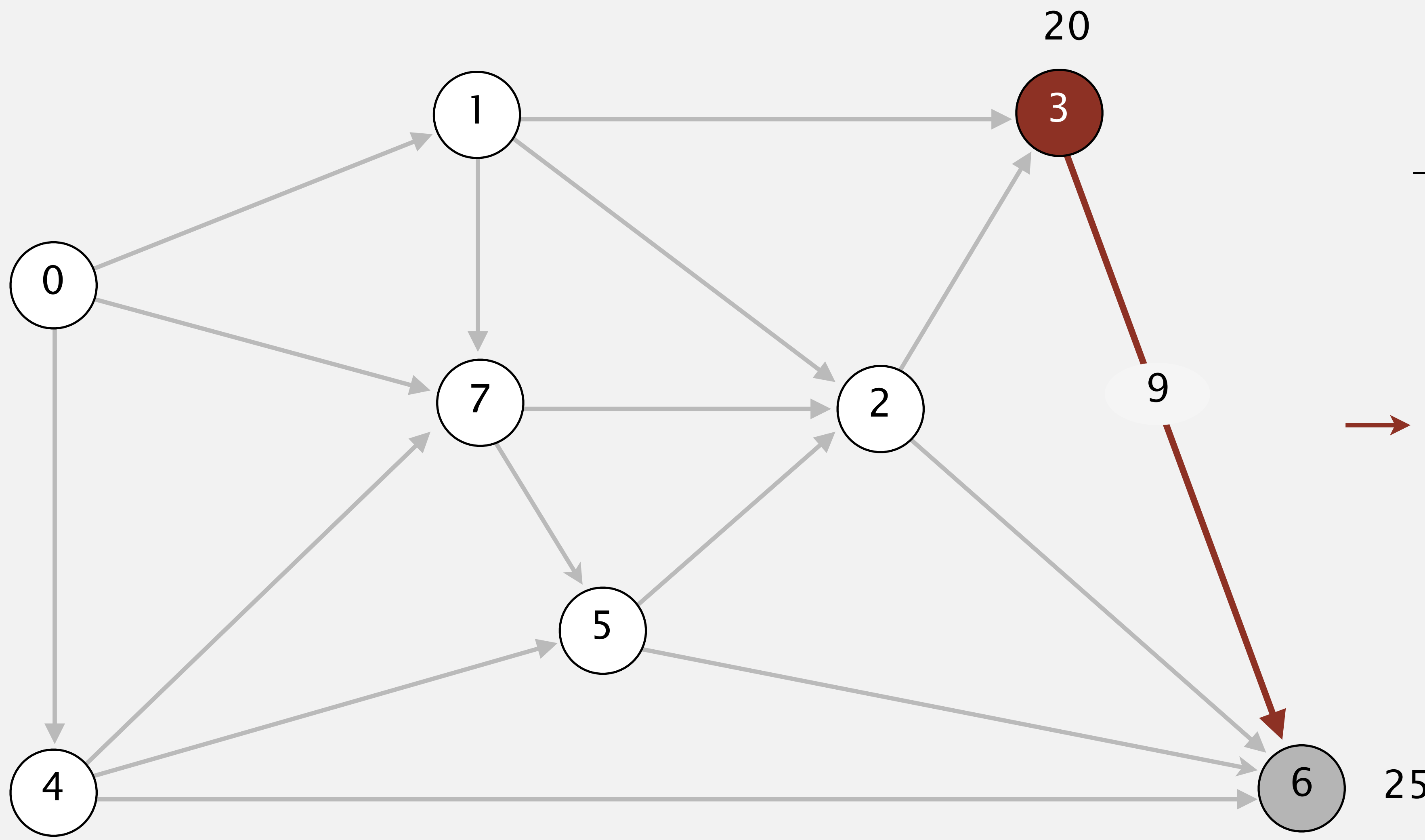


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

select vertex 3

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

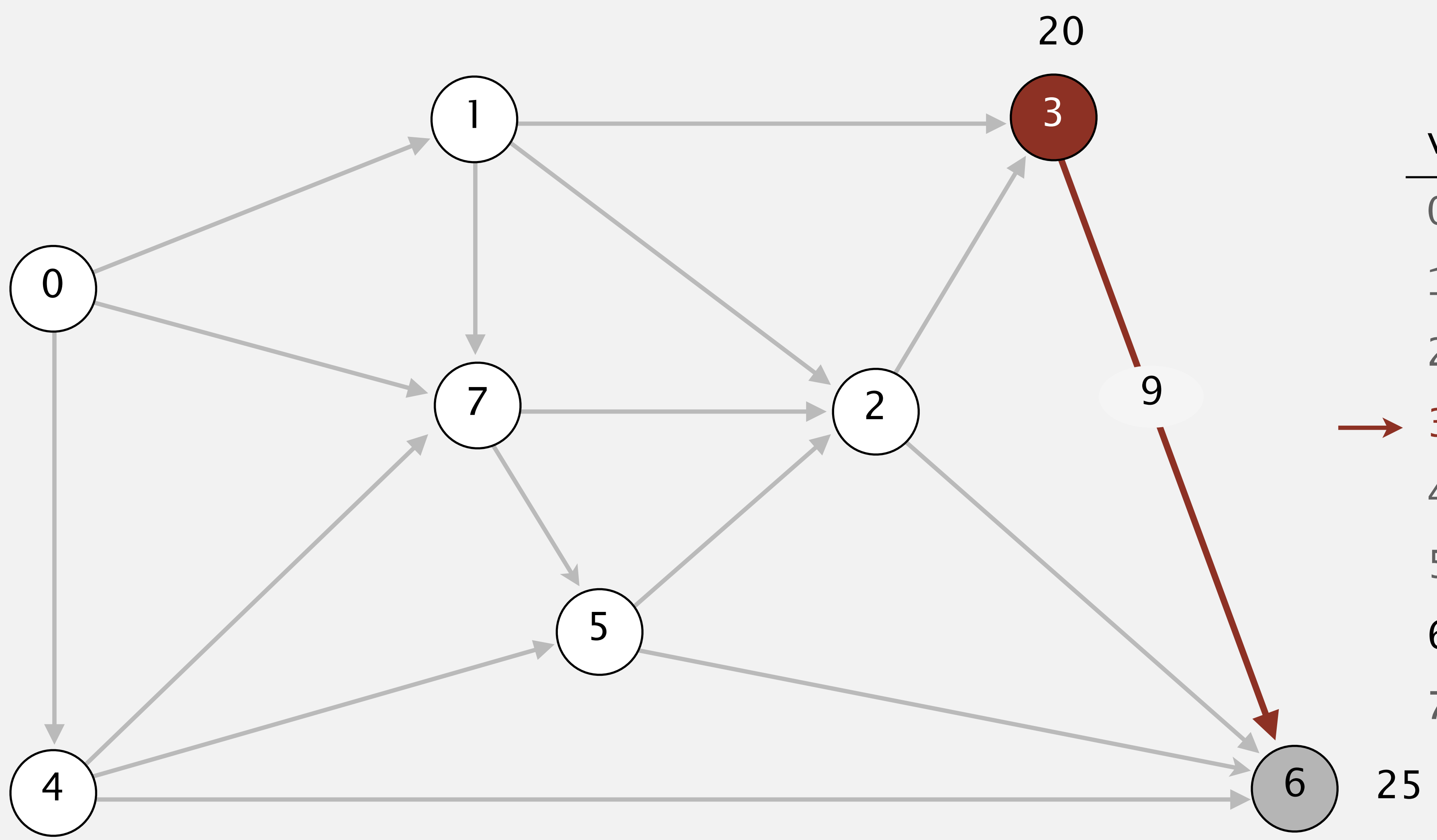


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

relax all edges pointing from 3

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

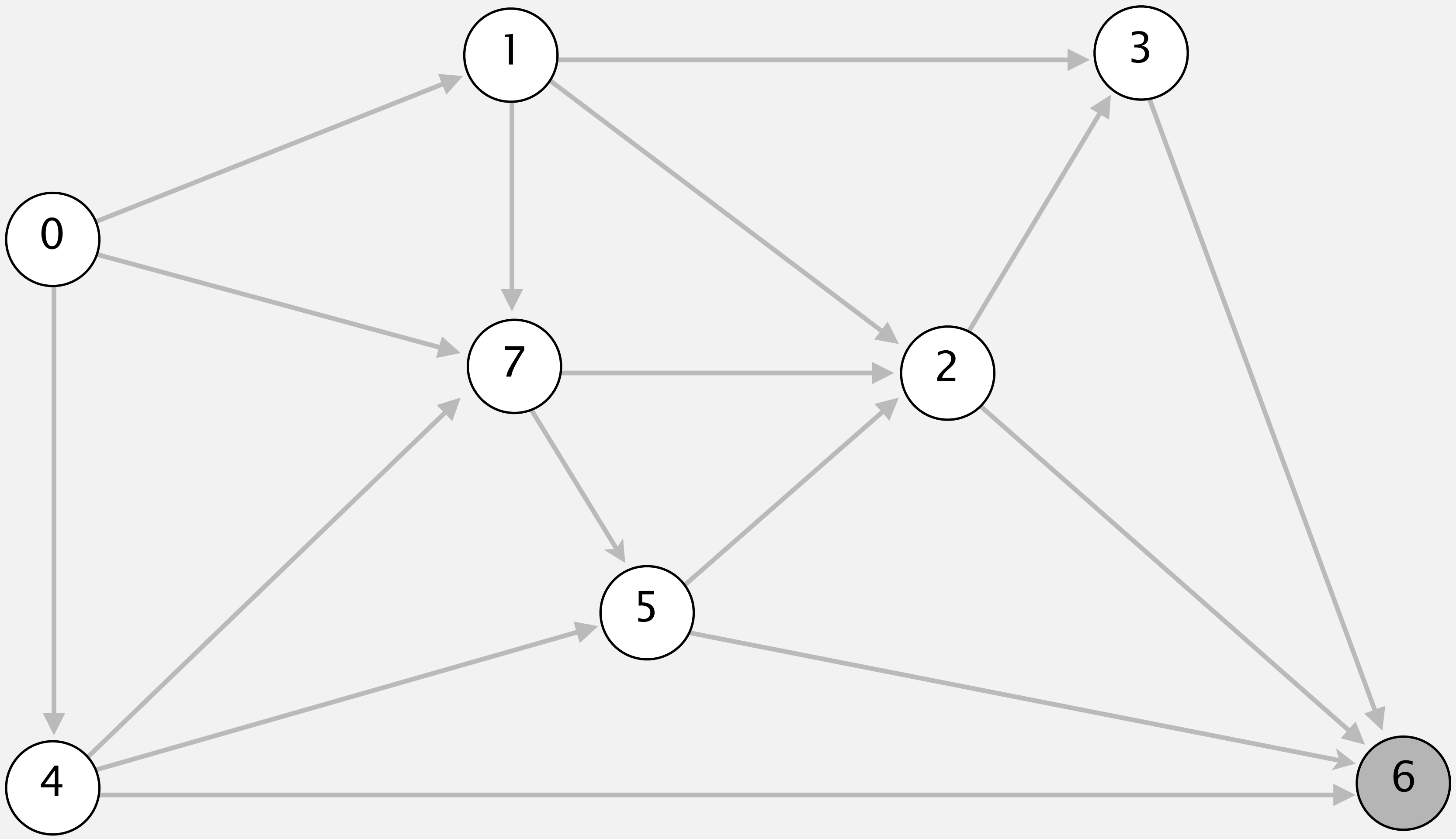


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0 ✔	2→6
7	8.0	0→7

relax all edges pointing from 3

Dijkstra's algorithm demo

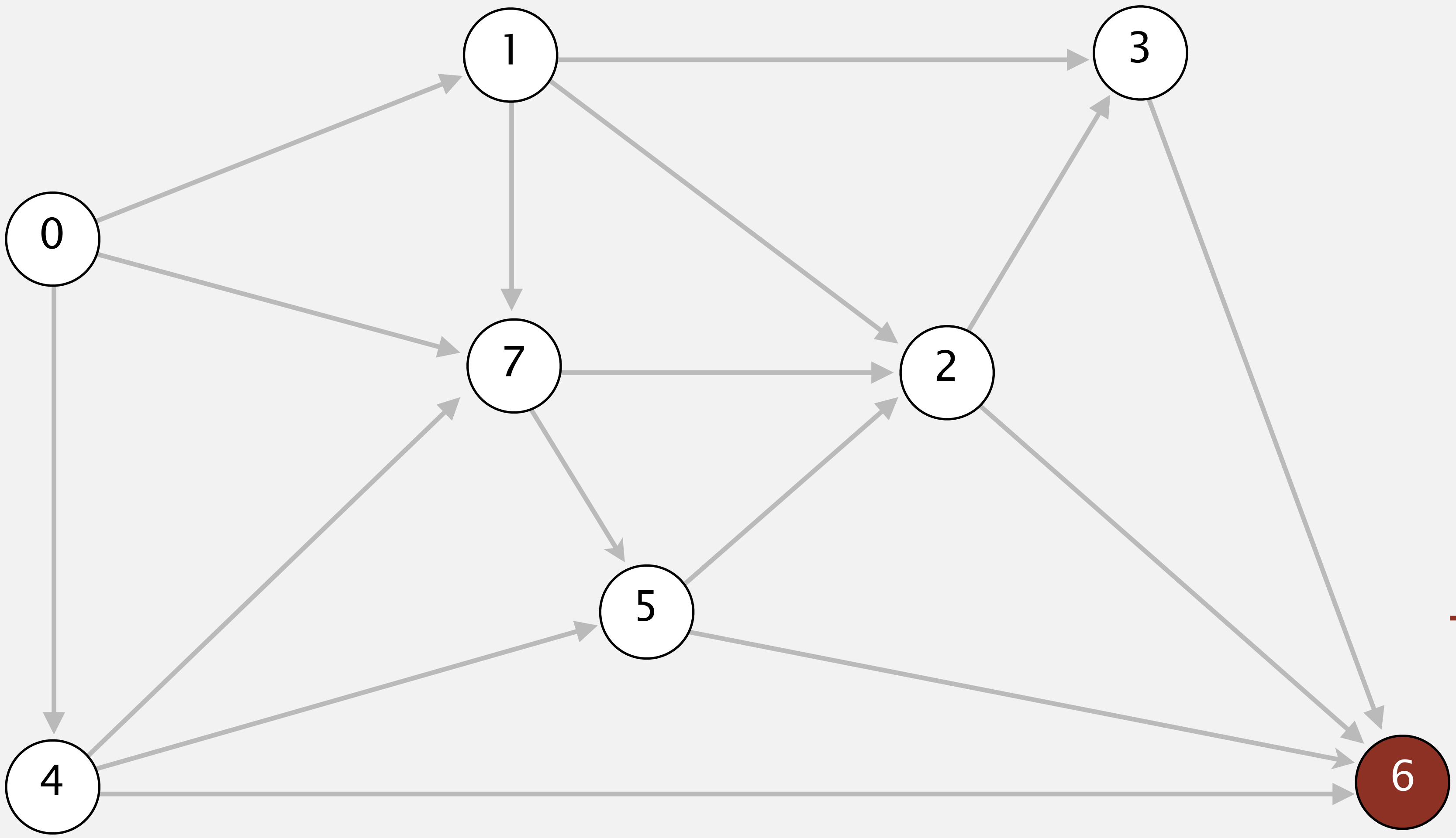
- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

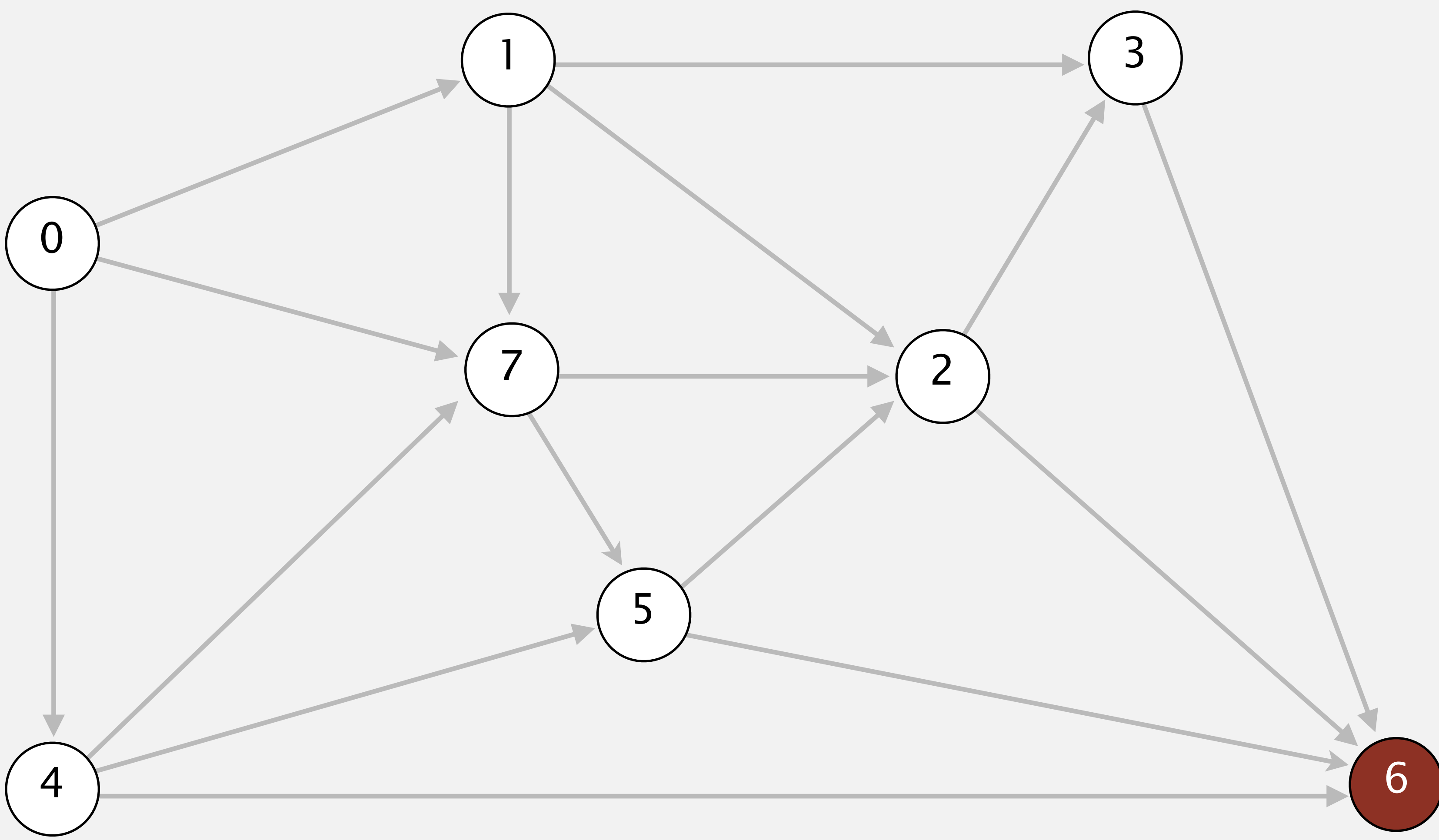


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

select vertex 6

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

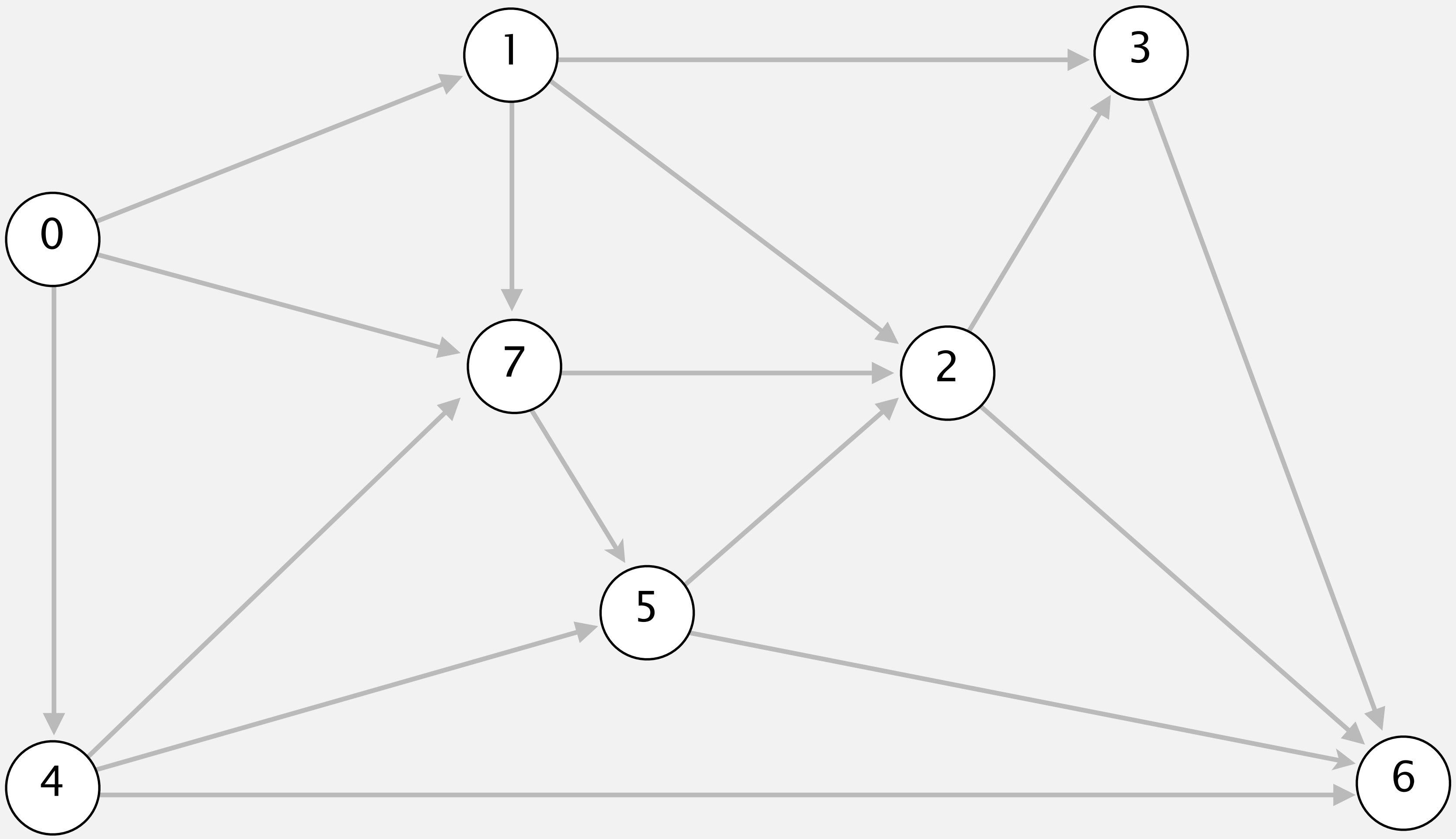


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

relax all edges pointing from 6

Dijkstra's algorithm demo

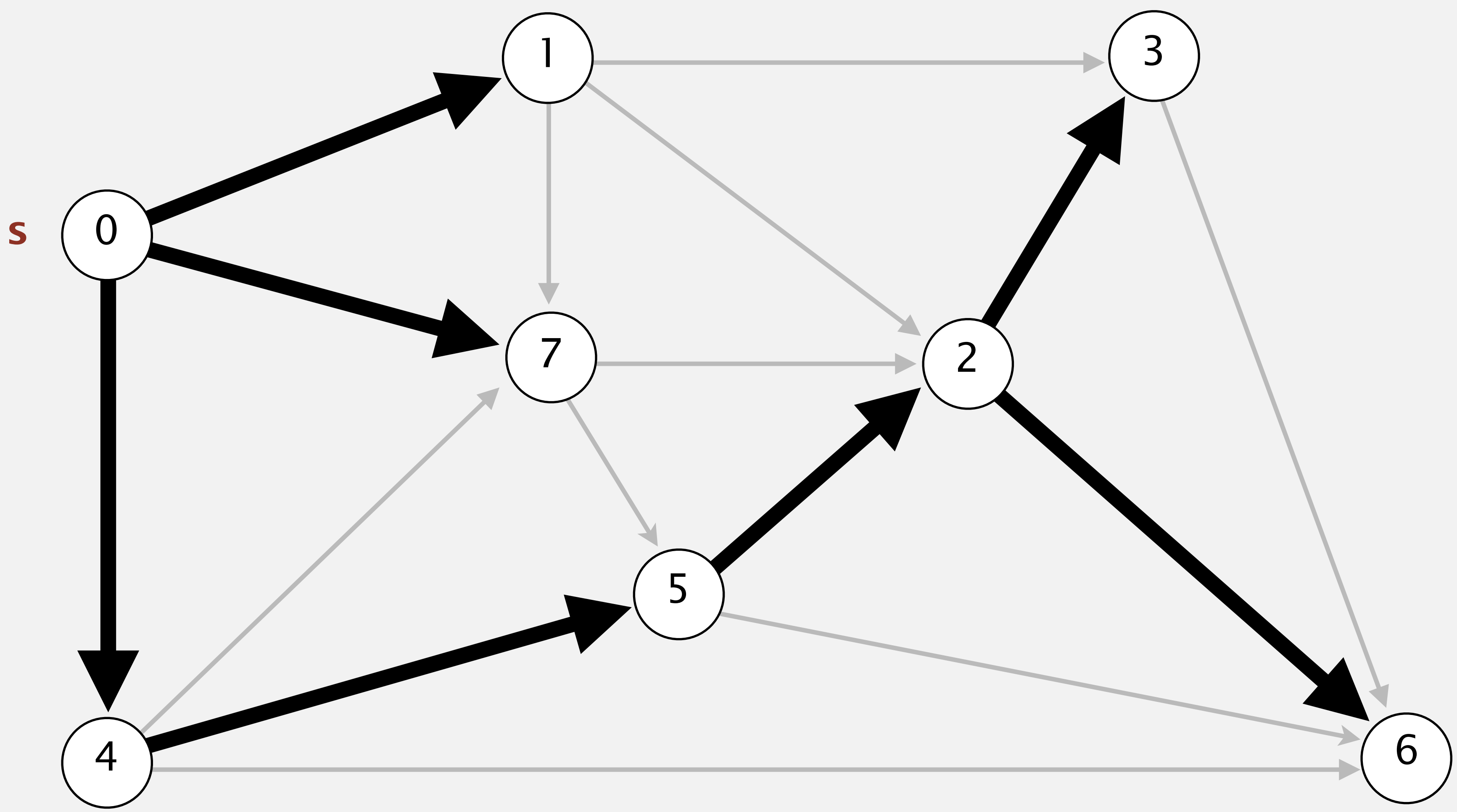
- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.

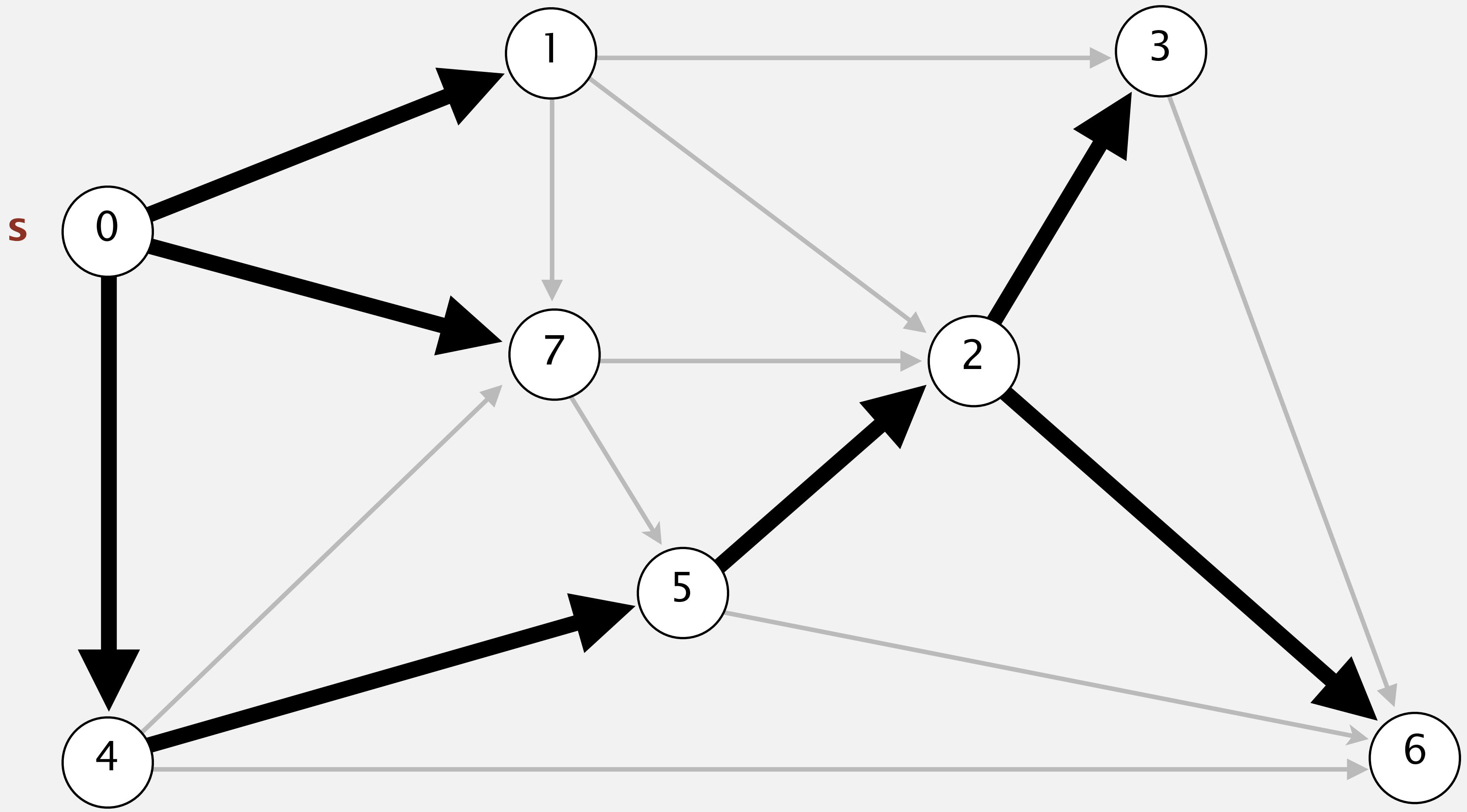


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

Dijkstra's algorithm demo

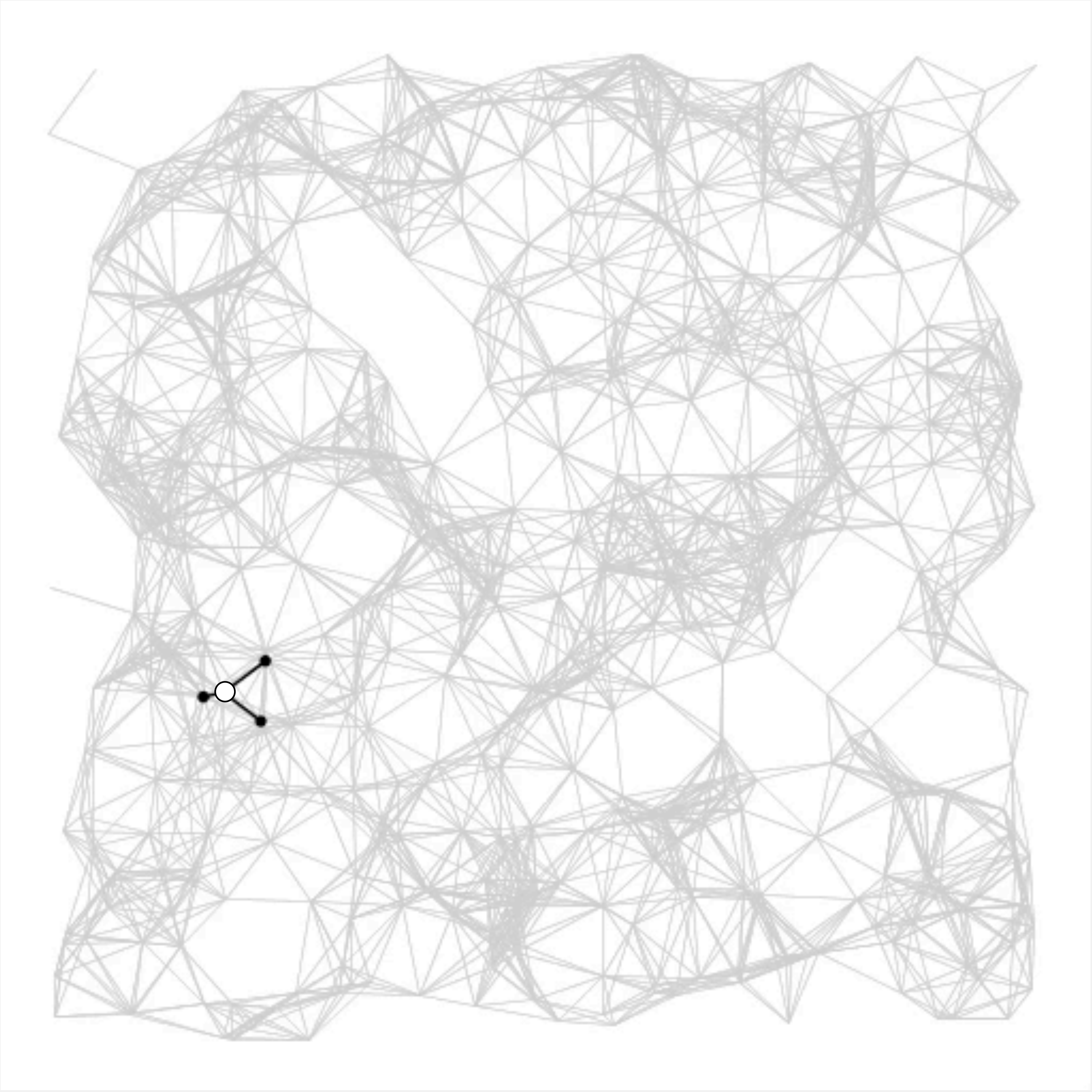
- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.



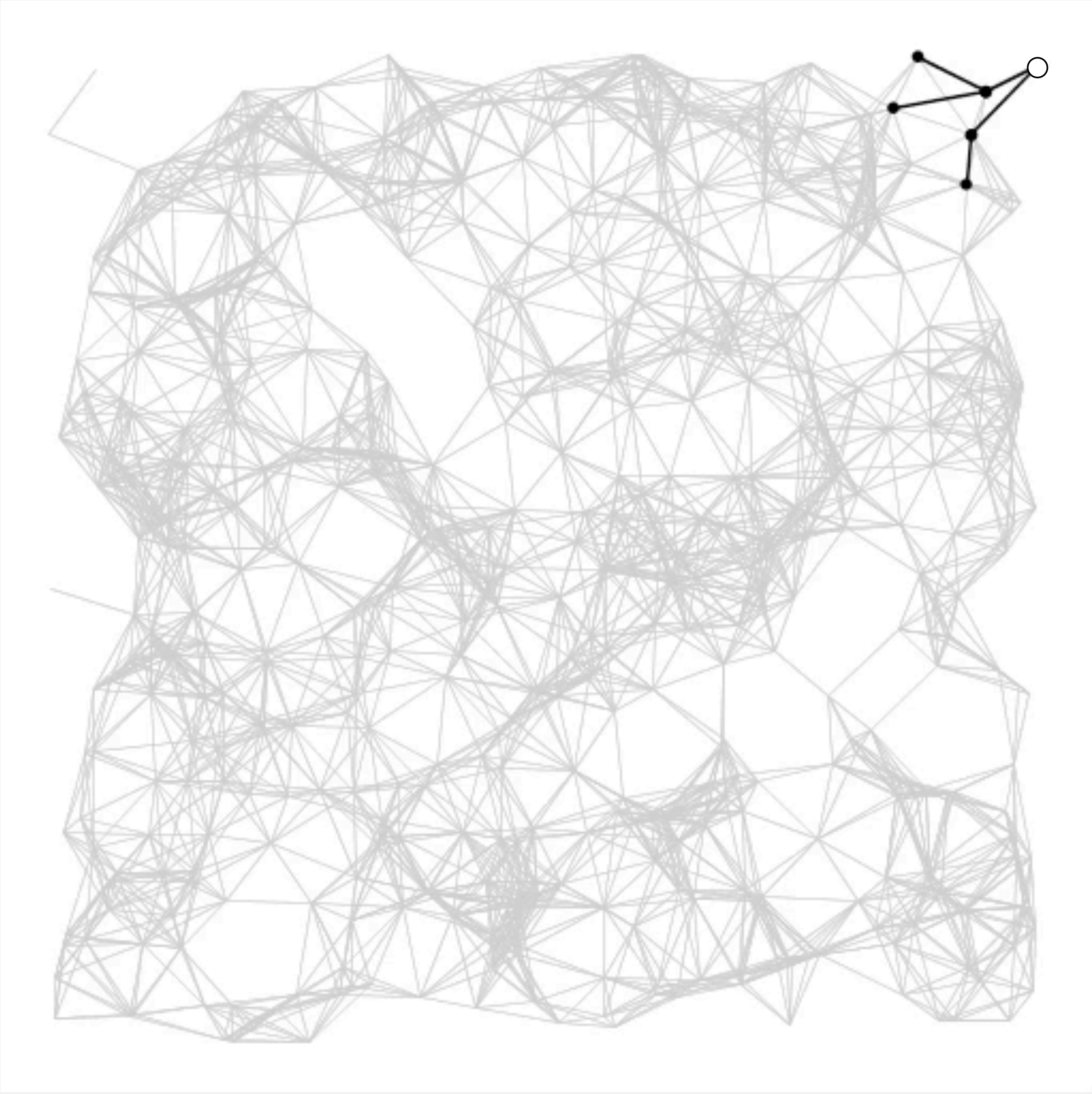
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

Dijkstra's algorithm visualization



Dijkstra's algorithm visualization



Dijkstra's algorithm: correctness proof 1

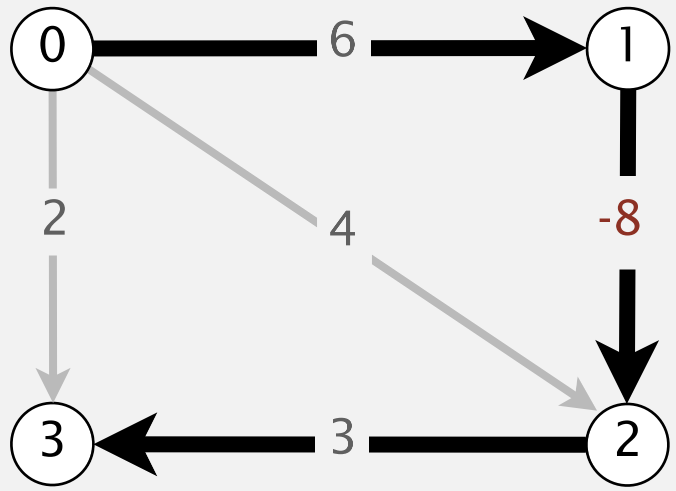
Proposition. Dijkstra's algorithm computes a SPT in any edge-weighted digraph with nonnegative weights.

Pf.

- Each edge e = v→w is relaxed exactly once (when vertex v is relaxed), leaving `distTo[w] ≤ distTo[v] + e.weight()`.
- Inequality holds until algorithm terminates because:
  - `distTo[w]` cannot increase ← `distTo[]` values are monotone decreasing
  - `distTo[v]` will not change ← we choose lowest `distTo[]` value at each step (and edge weights are nonnegative)
- Thus, upon termination, shortest-paths optimality conditions hold. ▪

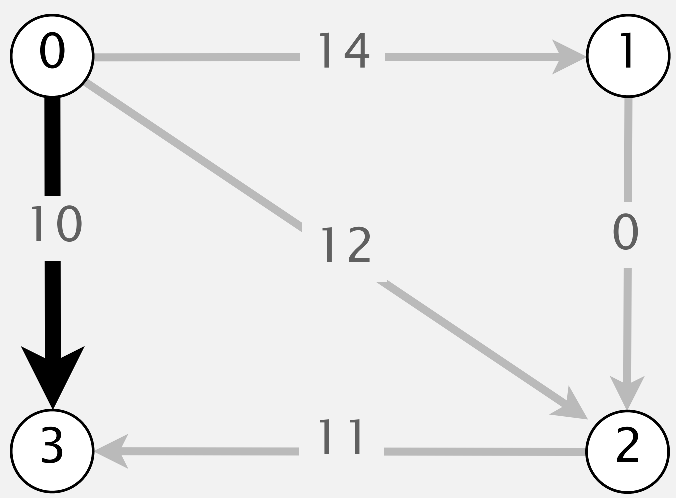
Shortest paths with negative weights: failed attempts

Dijkstra. Doesn't work with negative edge weights.



Dijkstra selects vertex 3 immediately after 0.
But shortest path from 0 to 3 is 0→1→2→3.

Re-weighting. Add a constant to every edge weight doesn't work.



Adding 8 to each edge weight changes the shortest path from 0→1→2→3 to 0→3.

Conclusion. Need a different algorithm.

Dijkstra's algorithm: Java implementation

```
public class DijkstraSP
{
   private DirectedEdge[] edgeTo;
   private double[] distTo;
   private IndexMinPQ<Double> pq;

   public DijkstraSP(EdgeWeightedDigraph G, int s)
   {
      edgeTo = new DirectedEdge[G.V()];
      distTo = new double[G.V()];
      pq = new IndexMinPQ<Double>(G.V());

      for (int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      pq.insert(s, 0.0);
      while (!pq.isEmpty())
      {
          int v = pq.delMin();
          for (DirectedEdge e : G.adj(v))
             relax(e);
      }
   }
}
```

relax vertices in order of distance from s

Dijkstra's algorithm: Java implementation

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
       if (pq.contains(w)) pq.decreaseKey(w, distTo[w]);
       else                pq.insert     (w, distTo[w]);
   }
}
```

update PQ

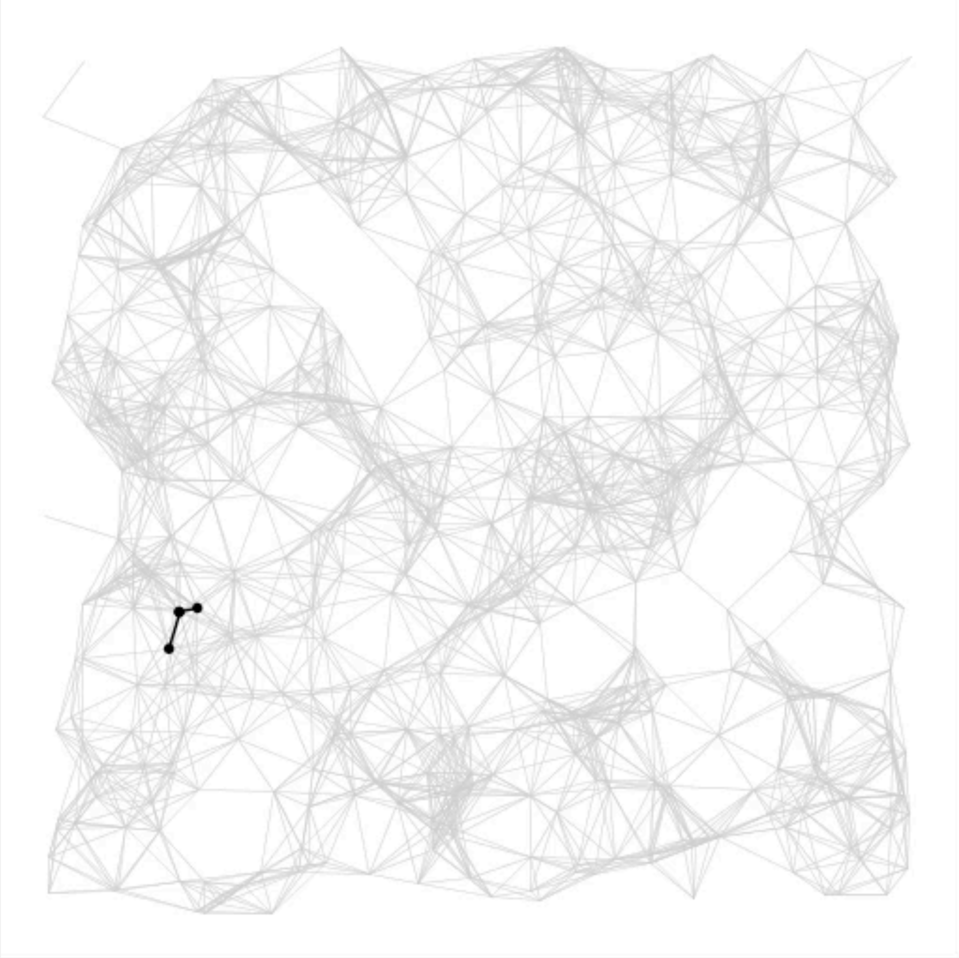
Computing a spanning tree in a graph

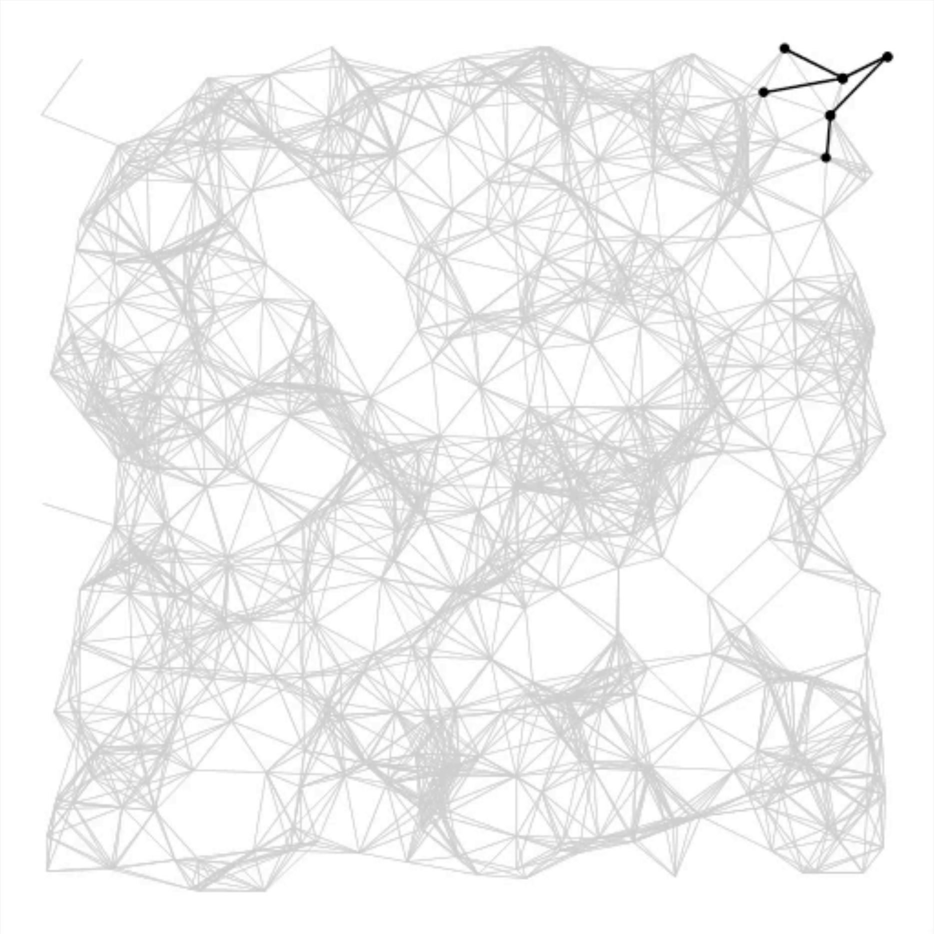
Dijkstra's algorithm seem familiar?

- Prim's algorithm is essentially the same algorithm.
- Both are in a family of algorithms that compute a spanning tree.

Main distinction: Rule used to choose next vertex for the tree.

- Prim: Closest vertex to the tree (via an undirected edge).
- Dijkstra: Closest vertex to the source (via a directed path).





Priority-first search

Insight. Four of our graph-search methods are the same algorithm!

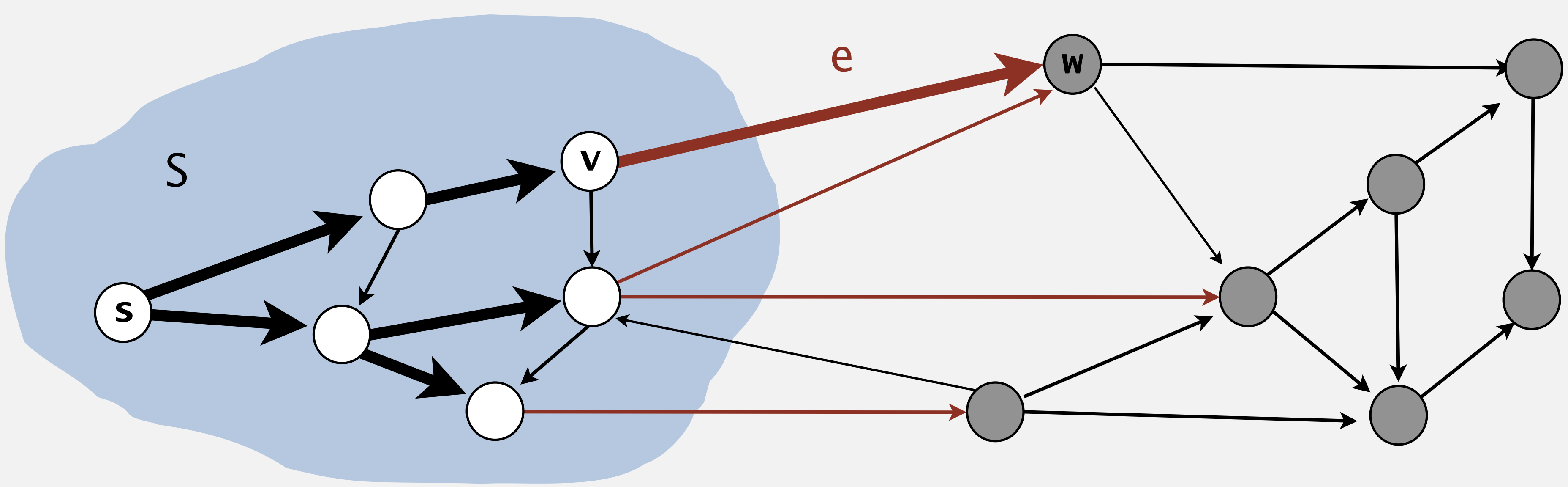
- Maintain a set of explored vertices S.
- Grow S by exploring edges with exactly one endpoint leaving S.

DFS. Take edge from vertex which was discovered most recently.

BFS. Take edge from vertex which was discovered least recently.

Prim. Take edge of minimum weight.

Dijkstra. Take edge to vertex that is closest to S.



Challenge. Express this insight in reusable Java code.

Dijkstra's algorithm: which priority queue?

Depends on PQ implementation: V insert, V delete-min, E decrease-key.

PQ implementation	insert	delete-min	decrease-key	total
unordered array	1	V	1	V^2
binary heap	$\log V$	$\log V$	$\log V$	$E \log V$
d-way heap	$\log_d V$	$d \log_d V$	$\log_d V$	$E \log_{E/V} V$

$E \log_D V$ where the best value for D is E/D

Bottom line.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.